GANITA PRAKASH

TEXTBOOK OF MATHEMATICS

GRADE

8

Part-I

0874

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NCERT

राष्ट्रीय शैक्षिक अनुसंधान और प्रशिक्षण परिषद्
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FOREWORD

The National Education Policy (NEP) 2020 envisages a system of education in the country that is rooted in an Indian ethos and its civilisational accomplishments in all fields of knowledge and human endeavour. At the same time, it aims to prepare students to engage constructively with the opportunities and challenges of the twenty-first century. The basis for this aspirational vision has been well laid out by the National Curriculum Framework for School Education (NCF-SE) 2023 across curricular areas at all stages. By nurturing students' inherent abilities across all five planes of human existence (*pañchakośhas*), the Foundational and Preparatory Stages set the Stage for further learning at the Middle Stage. Spanning Grades 6 to 8, the Middle Stage serves as a critical three-year bridge between the Preparatory and Secondary Stages.

The NCF-SE 2023, at the Middle Stage, aims to equip students with the skills that are needed to grow, as they advance in their lives. It aims to enhance their analytical, descriptive, and narrative capabilities, and to prepare them for the challenges and opportunities that await them. A diverse curriculum, covering nine subjects ranging from three languages—including at least two languages native to India—to Science, Mathematics, Social Sciences, Art Education, Physical Education and Well-being, and Vocational Education promotes their holistic development.

Such a transformative learning culture requires certain essential conditions. One of them is to have appropriate textbooks in different curricular areas, as these textbooks are intended to play a central role in mediating between content and pedagogy's role that helps strike a judicious balance between direct instruction and opportunities for exploration and inquiry. Among the other conditions, classroom arrangement and teacher preparation are crucial to establish conceptual connections both within and across curricular areas.

The National Council of Educational Research and Training (NCERT), on its part, is committed to providing students with such high-quality textbooks. Various Curricular Area Groups (CAGs), which have been constituted for this purpose, comprising notable subject-experts, pedagogues, and practising teachers as their members, have made all possible efforts to develop such textbooks. *Ganita Prakash*, the textbook of mathematics for Grade 8 (Part 1) is designed to make learning fun, engaging, and meaningful. Aligned with NEP 2020 and NCF-SE 2023, the textbook continues to foster the spirit of experiential and inquiry-based learning, a spirit already introduced in the mathematics textbooks for Grades 6 and 7. It encourages students to ask questions, think critically, and understand concepts of mathematics through real-world contexts.

The textbook makes efforts to encourage the students to observe and explore patterns around them and discover mathematical concepts on their own. The content is structured in a way that supports a joyful and progressive understanding of increasingly complex concepts, easing students into more advanced learning. The content attempts to integrate mathematics with other subject areas, such as Science, Social Science with cross-cutting themes like environmental education, value education, inclusive education, and Indian Knowledge Systems (IKS). At most of the places the concept begins with either a story or a puzzle that not only makes students think but also feels the importance of the concept.

However, in addition to this textbook, students at this stage should also be encouraged to explore various other learning resources. School libraries play a crucial role in making such resources available. Besides, the role of parents and teachers will also be invaluable in guiding and encouraging students to do so.

With this, I express my gratitude to all those who have been involved in the development of this textbook, and hope that it will meet the expectations of all stakeholders. At the same time, I also invite suggestions and feedback from all its users for further improvement in the coming years.

Dinesh Prasad Saklani
Director
National Council of Educational
Research and Training

New Delhi
June 2025

ABOUT THE BOOK

Mathematics helps students develop not only basic arithmetic skills, but also the crucial capacities of logical reasoning, creative problem-solving, and clear and precise communication (both oral and written). Mathematical knowledge also plays a crucial role in understanding concepts in other school subjects, such as Science and Social Science, and even Arts, Physical Education, and Vocational Education. Learning Mathematics contributes to the development of capacities to make informed choices and decisions. Understanding numbers and quantitative arguments is necessary for effective and meaningful democratic and economic participation. Mathematics thus plays an important role in achieving the overall aims of school education.

Mathematics at the Middle Stage is a major challenge and performs the dual role of being both close to the experience and environment of a learner and being abstract. It performs the role of developing intuition, while also maintaining and emphasising rigour. It is enhancing critical and logical thinking, while also developing artistry and creativity, and a sense of elegance and aesthetics. Mathematics provides students with plenty of opportunities to explore and discover concepts on their own while also teaching the best-known methods in the global repertoire of mathematics.

This textbook attempts to address the goals and challenges of learning mathematics. The writers have aimed to strike a judicious balance between informal and formal definitions and methods in helping students to develop both intuition and rigour. The textbook also provides numerous opportunities for student-student and student–teacher interaction in the classroom to promote active and experiential learning. A number of questions, puzzles, and interactive exercises are posed throughout the textbook to encourage constant exploration. Many of the questions are open-ended to stimulate classroom discussion.

The Chapter 1: 'A Square and A Cube' is an introduction to the special types of numbers called squares and cubes through engaging explorations, and contexts. Chapter 2: 'Power Play' considers short and logical ways of expressing very large numbers, and discusses how they can be written and read without ambiguity. Chapter 3: 'A Story of Numbers' reveals how the idea of number and number representation evolved over the course of time, across different places in the world, to finally reach its modern efficient form. Chapter 4: 'Quadrilaterals' discusses some interesting types of four-sided figures and problems based on them. Chapter 5: 'Number Play' shows different properties of numbers studied by the students with a balance of engaging activities through visualisation and rigorous mathematical reasoning. Chapter 6: 'We Distribute, Yet Things Multiply' covers aspects related to Algebra, and in particular, the distributive law. Chapter 7: 'Proportional Reasoning' explores a new way to compare two quantities using daily life situations. In all the chapters, an attempt has been made to emphasise connections with other subjects including Art, History and Science.

By weaving storytelling and hands-on activities together, as done in Grades 6 and 7, we hope that an immersive learning experience will be created that ignites curiosity and fosters a love for mathematics. It is hoped that teachers would give students the opportunity to discuss, play, engage with each other and provide logical arguments for different ideas, and find loopholes in arguments presented. This is necessary for the learners to eventually develop the ability to understand what it means to prove something and also become confident about the underlying concepts. The mathematics classroom should not expect a blind application of algorithms but should rather encourage students to find many different ways to solve problems.

As per the National Education Policy (NEP) 2020, computational thinking has also been gently introduced through puzzles, games, and interactive exercises that encourage students on these aspects. Indian rootedness has also been kept in mind while giving contexts for different concepts. The contributions of Indian mathematicians have also been given as part of a problem-solving approach to make students aware of India's rich mathematical heritage and its global contributions to mathematics.

The concepts and problems in this textbook are related to daily life situations. An attempt has been made to use contexts and materials that students are familiar with. Learning material sheets have been given at the back of the book that may be photocopied and used in the classroom. Exercises or activities are often given to encourage peer group efforts and discussions. However, this textbook intends to address the learning needs of a diverse group of students in the classroom.

We have tried to link concepts learnt in initial chapters with ideas in subsequent chapters, to show the connectedness and unity of mathematics. We hope that teachers can revise these concepts in a spiralling way so that students are able to appreciate the entire conceptual structure of mathematics. We hope that teachers may focus more on the ideas of squares and cubes, exponents, evolution of numbers and other notions that are new to the students, and are foundational for further learning.

Finally, this textbook aims to be more than just a textbook—it is a passport to a world of mathematical discovery and exploration. Whether used in the classroom or at home, we hope that it inspires students to embark on their own mathematical adventures, empowering them to see the beauty and relevance of Mathematics in everything. With its engaging approach and comprehensive coverage of Grade 8 mathematics concepts, this textbook aims to captivate young minds and set them on a lifelong journey of mathematical discovery.

I once again thank all the writers of and contributors of this textbook for their valuable contribution and service to the nation's mathematics teachers, learners and enthusiasts.

We look forward to your comments and suggestions regarding the textbook and hope that you will send interesting exercises, activities and tasks that you may develop while teaching and learning for future editions.

Ashutosh Wazalwar
Professor and *Academic Convener*
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NOTE TO THE TEACHER

We hope that this book, *Ganita Prakash*, will serve as a strong support and guide to you in achieving the exciting task that you have before you: that of passing on the joy of learning the beautiful subject of Mathematics to the next generation.

This task calls for providing a fertile environment that allows for the flowering of mathematical thinking in the minds of students. Classrooms, where students just listen and write down whatever is being told to them or written on the board, are deficient in the conditions required for learning mathematics. Instead, classrooms need to be places where students are engaged in playing with mathematical concepts, finding and discussing patterns, and developing creative strategies together to solve problems. Students should also be posing problems to each other and discussing possible solutions with each other. In fact, these are the very conditions that have led to the development of the entire field of mathematics so far, and so one cannot expect students to pick up mathematical thinking and understanding without these conditions.

Fortunately, it is not difficult to create such conditions in the classroom. It just requires an interesting question, problem, pattern, or challenge to be thrown open to the students on a regular basis, and sufficient time to be given to them to play with, discuss, and work on it as a class or in pairs or groups.

Along with it, an environment that accepts mistakes and acknowledges their importance in learning needs to be nurtured.

While creating the spark for initiating mathematical thinking in classrooms is not difficult, sustaining it may be challenging and may involve efforts from your side. Nevertheless, even if just the first part of throwing open a question, problem, pattern, or challenge is done at least once or twice a week, accompanied by sufficient waiting time from your side for students to play, discuss, and work on it, it can have a great positive impact on how the students view and approach mathematics.

It should be noted that this positive impact will not happen overnight. That takes time and depends on various factors such as the number of opportunities you give for problem solving, your patience, and the encouragement you give to the students.

To support you in posing problems, all the problems or questions in this book are marked using the icons ? and ?. These icons are indicators of potential opportunities to start off a process of problem-solving and exploration in the classroom. You will find some of the problems labelled 'Math Talk'. Such questions can especially be made as topics for classroom discussion.

An owl mascot appears at various points in the textbook to highlight important mathematical processes, ways of thinking, and problem-solving approaches. These can be brought out during classroom discussions, both where the owl is present and also in other similar situations.

In this grade, justification/proofs of mathematical statements find an increased presence. Students should be gently encouraged to deduce properties and not be forced to do it. Whenever students face challenges in doing it, encourage them to experiment and observe, and use their intuition in figuring out properties. Providing justification/proof is a skill that takes time to develop.

To develop students' mathematical thinking and understanding of concepts, a sufficient number of problems are given. Trying to 'cover' all of them must not happen at the cost of students not getting to spend quality time on playing with and discussing them.

It is important to understand that the exploratory problems are not only for promoting problem solving skills; they also serve in strengthening procedural fluency when children start engaging in exploration.

Efforts must be made in making students independent learners. One essential aspect required for this is an ability to read and understand mathematical text. To promote this skill, students should be encouraged to read the book by themselves and in groups. Give opportunities to them to interpret what they read and express it to others. This will also address the big problem that students face in speaking mathematics and interpreting word problems.

This textbook contains a number of open-ended problems. It also contains new treatments of certain concepts. If you are not able to solve them or follow some of them immediately, it is perfectly okay! Not everyone knows everything. Along with trying to understand and reflect upon such content,it will be very useful to take it to the classroom and open it up for discussion. After the discussion, things that are clear and those that are not yet clear can be clearly summarised. This process itself can throw a lot of light on the content.

In these discussions, you can participate as a fellow seeker, and when students see a teacher seek and think to understand something, it sets a wonderful example for them.

It is hoped that you and your students will have a great and fruitful time using this textbook!

Summary of Key Points

Time for Exploration

1. It is important to routinely pose new problems, questions, patterns, or challenges to the students and give them sufficient time to play with, discuss, and work on them, individually and in groups.
2. During this time, an environment that accepts mistakes and acknowledges their importance in learning needs to be nurtured.
3. There should be a culture where students pose problems to each other and discuss with each other various ways to approach the problems.

About the Problems in the Book

1. The exploratory problems in the book not only promote problem solving; they also aim to strengthen procedural fluency when students start engaging in exploration.
2. Trying to 'cover' all the problems in the book must not happen at the cost of students not getting to spend quality time on playing with, discussing, and solving them.

Reading

1. Encourage students to read the book by themselves and in groups.
2. Give opportunities to them to interpret what they read and to express it to others.

Right of Not Knowing!

1. It is perfectly okay if some of the content is not understood immediately. Along with trying to understand and reflect upon such content, it can also be taken to the classroom and opened up for discussion. After the discussion, things that are clear and those that are not yet clear can be clearly summarised. In these discussions, you can participate as a fellow seeker, and when students see a teacher seek and think to understand something, it sets a wonderful example for them!
2. Learning is a continual process. Indeed, there is so much in mathematics that is still not known and requires further exploration!

A NOTE TO STUDENTS!

To be able to appreciate the art of mathematics, it is not enough to just be a passive spectator. You need to immerse yourself in its process like a detective getting into action to solve a mystery.

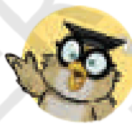
This is especially required when you see a new question or when a question arises from your own sense of wonder, or when you come across a new beautiful pattern. When you encounter these, pause your reading, and use your creativity to work out the question or understand and appreciate the pattern.

You will find that some questions are accompanied by their answers. Even if this is the case, it is worthwhile to work on the problems by yourself or in a group before you see the answer.

This will enrich your experience of going through the book!

Whenever there are questions coming up, you will see the icon ? and ?. This indicates that it is time for figuring things out!

? indicates a main question and ? indicates a sub-question.

The icons for owls  and  suggest some important processes in the learning of mathematics.

Sometimes you will find many questions collected together in a single place under the title '**Figure it Out**'.

Some questions are marked Math Talk. These questions are meant to be discussed and worked out with your peers.

Finally, there are questions marked Try This. These questions demand more creativity to be answered, and therefore will also often be more fun to answer as a result!

CONTENTS

THE CONSTITUTION OF INDIA

PREAMBLE

WE, THE PEOPLE OF INDIA, having solemnly resolved to constitute India into a [1]**[SOVEREIGN SOCIALIST SECULAR DEMOCRATIC REPUBLIC]** and to secure to all its citizens :

JUSTICE, social, economic and political;

LIBERTY of thought, expression, belief, faith and worship;

EQUALITY of status and of opportunity; and to promote among them all

FRATERNITY assuring the dignity of the individual and the [2][unity and integrity of the Nation];

IN OUR CONSTITUENT ASSEMBLY this twenty-sixth day of November, 1949 do **HEREBY ADOPT, ENACT AND GIVE TO OURSELVES THIS CONSTITUTION.**

1. Subs. by the Constitution (Forty-second Amendment) Act, 1976, Sec.2, for "Sovereign Democratic Republic" (w.e.f. 3.1.1977)
2. Subs. by the Constitution (Forty-second Amendment) Act, 1976, Sec.2, for "Unity of the Nation" (w.e.f. 3.1.1977)

1 A SQUARE AND A CUBE

0874CH01

Queen Ratnamanjuri had a will written that described her fortune of *ratnas* (precious stones) and also included a puzzle. Her son Khoisnam and their 99 relatives were invited to the reading of her will. She wanted to leave all of her *ratnas* to her son, but she knew that if she did so, all their relatives would pester Khoisnam forever. She hoped that she had taught him everything he needed to know about solving puzzles. She left the following note in her will—

"I have created a puzzle. If all 100 of you answer it at the same time, you will share the *ratnas* equally. However, if you are the first one to solve the problem, you will get to keep the entire inheritance to yourself. Good luck."

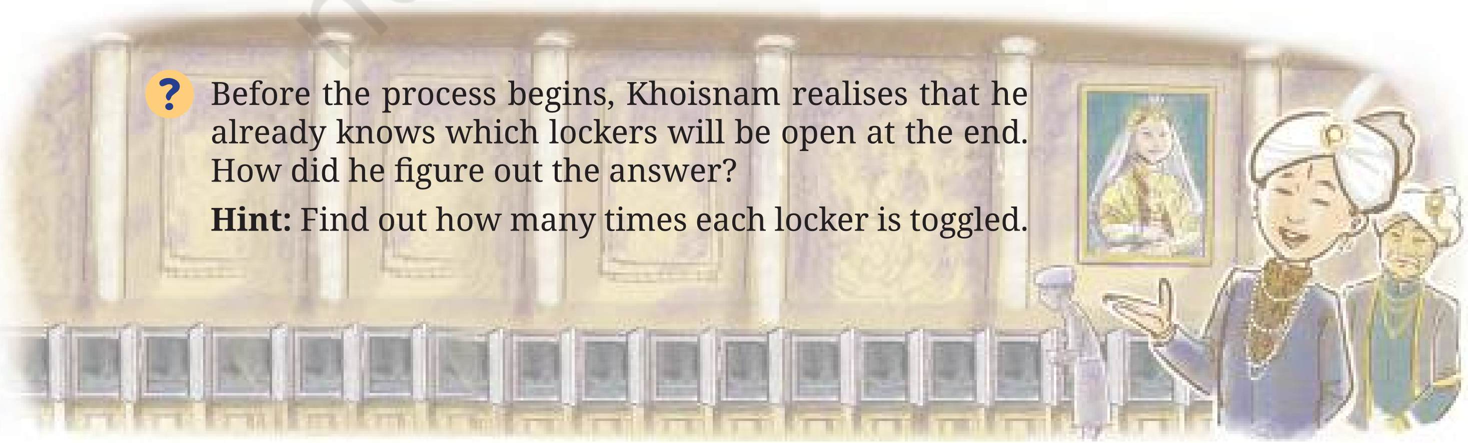
The minister took Khoisnam and his 99 relatives to a secret room in the mansion containing 100 lockers.

The minister explained— "Each person is assigned a number from 1 to 100.

- Person 1 opens every locker.
- Person 2 toggles every 2nd locker (i.e., closes it if it is open, opens it if it is closed).
- Person 3 toggles every 3rd locker (3rd, 6th, 9th, ... and so on).
- Person 4 toggles every 4th locker (4th, 8th, 12th, ... and so on).

This continues until all 100 get their turn.

In the end, only some lockers remain open. The open lockers reveal the code to the fortune in the safe."



? Before the process begins, Khoisnam realises that he already knows which lockers will be open at the end. How did he figure out the answer?

Hint: Find out how many times each locker is toggled.

If a locker is toggled an odd number of times, it will be open. Otherwise, it will be closed. The number of times a locker is toggled is the same as the number of factors of the locker number. For example, for locker #6, Person 1 opens it, Person 2 closes it, Person 3 opens it and Person 6 closes it. The numbers 1, 2, 3, and 6 are factors of 6. If the number of factors is even, the locker will be toggled by an even number of people and it will eventually be closed.

6:
1 × 6
2 × 3
Factors are
1, 2, 3 and 6.

Note that each factor of a number has a 'partner factor' so that the product of the pair of factors yields the given number. Here, 1 and 6 form a pair of partner factors of 6, and 2 and 3 form another pair.

? Does every number have an even number of factors?

1:
1 × 1
The only factor is 1.

4:
1 × 4
2 × 2
Factors are
1, 2 and 4.

9:
1 × 9
3 × 3
Factors are
1, 3 and 9.

We see in some cases, like 2 × 2, that the numbers in the pair are the same.

? Can you use this insight to find more numbers with an odd number of factors?

For instance, 36 has a factor pair 6 × 6 where both numbers are 6. Does this number have an odd number of factors? If every factor of 36 other than 6 has a different factor as its partner, then we can be sure that 36 has an odd number of factors. Check if this is true.

Hence all the following numbers have an odd number of factors —

$$1 \times 1, 2 \times 2, 3 \times 3, 4 \times 4, \ldots$$

A number that can be expressed as the product of a number with itself is called a **square number**, or simply a **square**. The only numbers that have an odd number of factors are the squares, because they each have one factor which, when multiplied by itself, equals the number. Therefore, every locker whose number is a square will remain open.

? Write the locker numbers that remain open.

Khoisnam immediately collects word clues from these 10 lockers and reads, "The passcode consists of the first five locker numbers that were touched exactly twice."

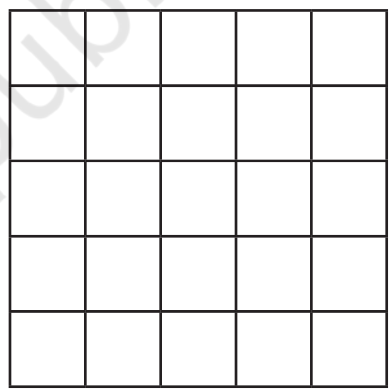
? Which are these five lockers?

The lockers that are toggled twice are the prime numbers, since each prime number has 1 and the number itself as factors. So, the code is 2-3-5-7-11.

1.1 Square Numbers

Why are the numbers, 1, 4, 9, 16, …, called squares? We know that the number of unit squares in a square (the area of a square) is the product of its sides. The table below gives the areas of squares with different sides.

Sidelength (in units)	Area (in sq units)
1	1 × 1 = 1 sq. unit
2	2 × 2 = 4 sq. units
3	3 × 3 = 9 sq. units
4	4 × 4 = 16 sq. units
5	5 × 5 = 25 sq. units
10	10 × 10 = 100 sq. units



We use the following notation for squares.

$$1 \times 1 = 1^2 = 1$$
$$2 \times 2 = 2^2 = 4$$
$$3 \times 3 = 3^2 = 9,$$
$$4 \times 4 = 4^2 = 16$$
$$5 \times 5 = 5^2 = 25.$$
$$\vdots$$

In general, for any number n, we write $n \times n = n^2$, which is read as 'n squared'.

Can we have a square of sidelength $\frac{3}{5}$ or 2.5 units?

Yes, there area in square units are $(\frac{3}{5})^2 = (\frac{3}{5}) \times (\frac{3}{5}) = (\frac{9}{25})$,

and $(2.5)^2 = (2.5) \times (2.5) = 6.25$.

The squares of natural numbers are called **perfect squares.** For example, 1, 4, 9, 16, 25, ... are all perfect squares.

Patterns and Properties of Perfect Squares

Find the squares of the first 30 natural numbers and fill in the table below.

$1^2 = 1$	$11^2 = 121$	$21^2 = 441$
$2^2 = 4$	$12^2 =$	$22^2 =$
$3^2 = 9$	$13^2 =$	
$4^2 = 16$	$14^2 =$	
$5^2 = 25$	$15^2 =$	
$6^2 =$	$16^2 =$	
$7^2 =$	$17^2 =$	
$8^2 =$	$18^2 =$	
$9^2 =$	$19^2 =$	
$10^2 =$	$20^2 =$	

? What patterns do you notice? Share your observations and make conjectures.

Math Talk

Study the squares in the table above. What are the digits in the units places of these numbers? All these numbers end with 0, 1, 4, 5, 6 or 9. None of them end with 2, 3, 7 or 8.

? If a number ends in 0, 1, 4, 5, 6 or 9, is it always a square?

Math Talk

The numbers 16 and 36 are both squares with 6 in the units place. However, 26, whose units digit is also 6, is not a square. Therefore, we cannot determine if a number is a square just by looking at the digit in the units place. But, the units digit can tell us when a number is not a square. If a number ends with 2, 3, 7, or 8, then we can definitely say that it is not a square.

? Write 5 numbers such that you can determine by looking at their units digit that they are not squares.

The squares, 1^2, 9^2, 11^2, 19^2, 21^2, and 29^2, all have 1 in their units place. Write the next two squares. Notice that if a number has 1 or 9 in the units place, then its square ends in 1.

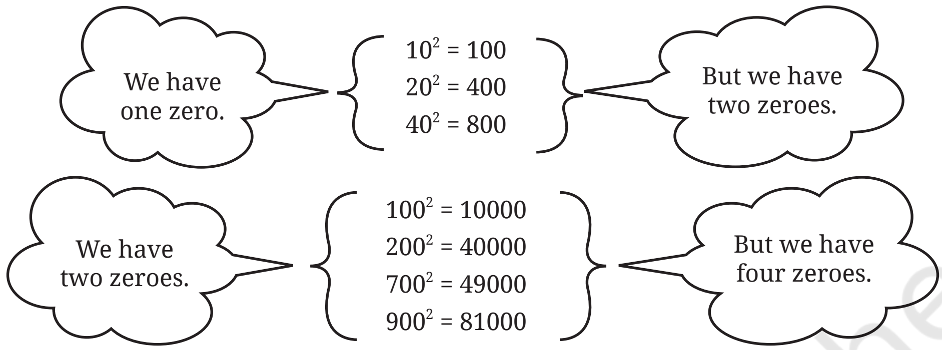
? Let us consider square numbers ending in 6: $16 = 4^2$, $36 = 6^2$, $196 = 14^2$, $256 = 16^2$, $576 = 24^2$, and $676 = 26^2$.

Which of the following numbers have the digit 6 in the units place?

(i) 38^2 (ii) 34^2 (iii) 46^2 (iv) 56^2 (v) 74^2 (vi) 82^2

? Find more such patterns by observing the numbers and their squares from the table you filled earlier.

Consider the following numbers and their squares.



? If a number contains 3 zeros at the end, how many zeros will its square have at the end?

? What do you notice about the number of zeros at the end of a number and the number of zeros at the end of its square? Will this always happen? Can we say that squares can only have an even number of zeros at the end?

? What can you say about the parity of a number and its square?

Perfect Squares and Odd Numbers

Let us explore the differences between consecutive squares. What do you notice?

$4 - 1 = \mathbf{3}$ $9 - 4 = \mathbf{5}$ $16 - 9 = \mathbf{7}$ $25 - 16 = \mathbf{9}$

See if this pattern continues for the next few square numbers.

From this we observe that adding consecutive odd numbers starting from 1 gives consecutive square numbers, as shown below.

$1 = \mathbf{1}$

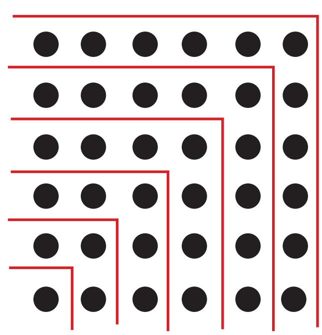
$1 + 3 = \mathbf{4}$

$1 + 3 + 5 = \mathbf{9}$

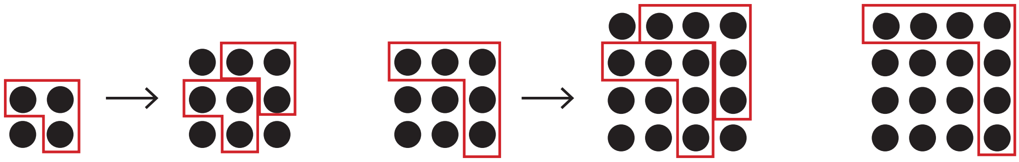
$1 + 3 + 5 + 7 = \mathbf{16}$

$1 + 3 + 5 + 7 + 9 = \mathbf{25}$

$1 + 3 + 5 + 7 + 9 + 11 = \mathbf{36}$.



Do you remember this pattern from Grade 6?
The picture below explains why each subsequent inverted L gives the next odd number:



$1 + 3$ $\quad$ $1 + 3 + (3 + 2)$ $\quad$ $1 + 3 + 5$ $\quad$ $1 + 3 + 5 + (5 + 2)$ $\quad$ $1 + 3 + 5 + 7$

We see that the sum of the first n odd numbers is n^2. Alternatively, every square is a sum of successive odd numbers starting from 1.

In mathematics, sometimes arguments and reasoning can be presented without any words. Visual proofs can be complete by themselves.

Also, we can find out whether a number is a perfect square by successively subtracting odd numbers. Consider the number 25, successively subtract 1, 3, 5, ... until you get or cross over 0,

$25 - 1 = 24 \quad 24 - 3 = 21 \quad 21 - 5 = 16 \quad 16 - 7 = 9 \quad 9 - 9 = 0$

This means $25 = 1 + 3 + 5 + 7 + 9$ and is thus a perfect square. Since we subtracted the first five odd numbers, $25 = 5^2$.

? Using the pattern above, find 36^2, given that $35^2 = 1225$.

From the question we know that 1225 is the sum of the first 35 odd numbers. To find 36^2, we need to add the 36th odd number to 1225.

? How do we find the 36th odd number?

The 1st odd number is 1, 2nd odd number is 3, 3rd number is 5, ..., 6th odd number is 11 and so on.

? What is the n^{th} odd number?

The n^{th} odd number is $2n–1$.
Therefore, the 36th odd number is 71.
By adding 71 to 1225, we get 1296, which is 36^2.

Consider a number such as 38 that is not a square and subtract consecutive odd numbers starting from 1.

$38 - 1 = 37 \quad 37 - 3 = 34 \quad 34 - 5 = 29 \quad 29 - 7 = 22 \quad 22 - 9 = 13$

$13 - 11 = 2 \quad 2 - 13 = - 11$

This shows that 38 cannot be expressed as a sum of consecutive odd numbers starting with 1.

Thus, we can say that a natural number is not a perfect square if it cannot be expressed as a sum of successive odd natural numbers starting from 1. We can use this result to find out whether a natural number is a perfect square.

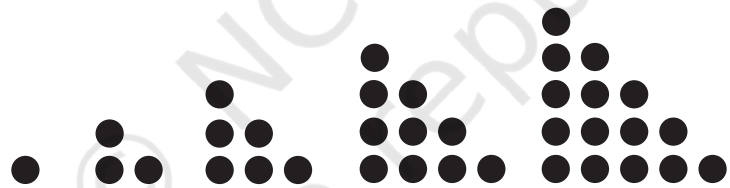
? Find how many numbers lie between two consecutive perfect squares. Do you notice a pattern?

? How many square numbers are there between 1 and 100? How many are between 101 and 200? Using the table of squares you filled earlier, enter the values below, tabulating the number of squares in each block of 100. What is the largest square less than 1000?

1 –100	101 – 200	201 – 300	301 – 400	401 – 500
____	____	____	____	____
501 – 600	601 – 700	701 – 800	801 – 900	901 – 1000
____	____	____	____	____

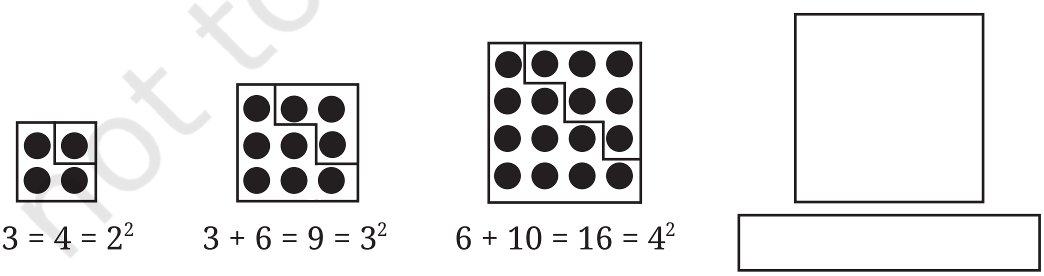
Perfect Squares and Triangular Numbers

Do you remember triangular numbers?



1 3 6 10 15

? Can you see any relation between triangular numbers and square numbers? Extend the pattern shown and draw the next term.



$1 + 3 = 4 = 2^2$ $3 + 6 = 9 = 3^2$ $6 + 10 = 16 = 4^2$

Square Roots

? The area of a square is 49 sq. cm. What is the length of its side?

We know that $7 \times 7 = 49$, or $7^2 = 49$.

So, the length of the side of a square with an area of 49 sq. cm is 7 cm. We call 7 the **square root** of 49.
In general, if $y = x^2$ then x is the **square root** of y.

? What is the square root of 64?

We know that 8 × 8 is 64. So, 8 is the square root of 64. What about – 8 × – 8 ? That is 64 too!

$$8^2 = 64, \text{ and } (-8)^2 = 64.$$

So, the square roots of 64 are + 8 and – 8.

Every perfect square has two integer square roots. One is positive and the other is negative. The square root of a number is denoted by $\sqrt{\ }$

Thus, $\sqrt{64} = \pm 8$ and $\sqrt{100} = \pm 10$.

Note that $\sqrt{8^2} = \pm 8$ and $\sqrt{10^2} = \pm 10$. In general, $\sqrt{n^2} = \pm n$.
In this chapter, we shall only consider the positive square root.

? Given a number, such as 576 or 327, how do we find out if it is a perfect square? If it is a perfect square, how can we find its square root?

Math Talk

We know that perfect squares end in 1, 4, 9, 6, 5, or an even number of zeros. But, it is not certain that a number that satisfies this condition is a square.

We can clearly say that 327 is not a perfect square. However, we cannot be sure that 576 is a perfect square.

1. We can list all the square numbers in sequence and find out whether 576 occurs among them. We know that 202 = 400, we can find squares of 21, 22, 23, ... and so on until we get 576 or a number greater than 576.

   $20^2 = 400$ $21^2 = 441$ $22^2 = 484$ $23^2 = 529$ $24^2 = 576$

   However, this process becomes inefficient for larger numbers.

2. Recall that every square can be expressed as a sum of consecutive odd numbers starting from 1.

   Consider $\sqrt{81}$.

   81 – 1 = 80   80 – 3 = 77   77 – 5 = 72   72 – 7 = 65   65 – 9 = 56

   56 – 11 = 45   45 – 13 = 32   32 – 15 = 17   17 – 17 = 0

   From 81, we successively subtracted consecutive odd numbers starting from 1 until we obtained 0 at the 9th step. Therefore $\sqrt{81}$ = 9.

   Can we find the square root of 729 using this method? Yes, but it will be time-consuming.

3. We know that a perfect square is obtained by multiplying an integer by itself. Will looking at a number's prime factorisation help in determining whether it is a perfect square?

   Yes, if we can divide the prime factors of a number into two equal groups, then the product of the prime factors in either group combine to form the square root.

? Is 324 a perfect square?

$$324 = 2 \times 2 \times 3 \times 3 \times 3 \times 3.$$

These can be grouped as

$$324 = (2 \times 3 \times 3) \times (2 \times 3 \times 3).$$

$$= (2 \times 3 \times 3)^2 = 18^2.$$

We can also write the prime factors in pairs. That is,

$$324 = (2 \times 2) \times (3 \times 3) \times (3 \times 3),$$

which shows that 324 is a perfect square. Thus,

$$324 = (2\times3\times3)^2 = 18^2.$$

Therefore, $\sqrt{324} = 18$.

? Is 156 a perfect square?

The prime factorisation of 156 is $2 \times 2 \times 3 \times 13$.
We cannot pair up these factors.
Therefore, 156 is not a perfect square.

? Find whether 1156 and 2800 are perfect squares using prime factorisation.

We can estimate the square root of larger perfect squares by looking at the closest perfect squares we are familiar with and then narrowing down the interval to search.

For example, to find $\sqrt{1936}$, we can reason as follows:

(i) 1936 is between 1600 (40^2) and 2500 (50^2), so $40 < \sqrt{1936} < 50$.

(ii) The last digit of 1936 is 6. So, the last digit of the square root must either be 4 or 6. It can be 44 or 46.

(iii) If we calculate 45^2, we can compare it with 1936 to halve the interval to search from 40–50 to either 40–45 or 45–50. We can write 45^2 as $(40 + 5)(40 + 5) = 40^2 + 2 \times 40 \times 5 + 5^2 = 1600 + 400 + 25 = 2025$.

(iv) $2025 > 1936$. So, $40 < \sqrt{1936} < 45$

(v) From the observation in point b we can guess and then verify that $\sqrt{1936}$ is 44.

Consider the following situations —

Aribam and Bijou play a game. One says a number and the other replies with its square root. Aribam starts. He says 25, and Bijou quickly responds with 5. Then Bijou says 81, and Aribam answers 9. The game goes on till Aribam says 250. Bijou is not able to answer because 250 is not a perfect square. Aribam asks Bijou if he can at least provide a number that is close to the square root of 250.

For this, Bijou needs to estimate the square root of 250.
We know that 100 < 250 < 400 and $\sqrt{100} = 10$ and $\sqrt{400} = 20$.
So, $10 < \sqrt{250} < 20$.

But, we are still not very close to the number whose square is 250.
We know that $15^2 = 225$ and $16^2 = 256$.

Therefore, $15 < \sqrt{250} < 16$. Since 256 is much closer to 250 than 225, $\sqrt{250}$ is approximately 16. We also know it is less than 16.

Here is another problem that requires estimating square roots.

Akhil has a square piece of cloth of area 125 cm^2. He wants to know if he can cut out a square handkerchief of side 15 cm. If not, he wants to know the maximum size handkerchief that can be cut out from this piece of cloth with an integer side length.

125 is not a perfect square. The nearest perfect squares are $11^2 = 121$ and $12^2 = 144$. So the largest square handkerchief with integer side length that can be cut out from this piece of cloth has side length 11 cm.

? Figure it Out

1. Which of the following numbers are not perfect squares?
   (i) 2032 (ii) 2048 (iii) 1027 (iv) 1089
2. Which one among 64^2, 108^2, 292^2, 36^2 has last digit 4?
3. Given $125^2 = 15625$, what is the value of 126^2?
   (i) 15625 + 126 (ii) $15625 + 26^2$ (iii) 15625 + 253
   (iv) 15625 + 251 (v) $15625 + 51^2$
4. Find the length of the side of a square whose area is 441 m^2.
5. Find the smallest square number that is divisible by each of the following numbers: 4, 9, and 10.
6. Find the smallest number by which 9408 must be multiplied so that the product is a perfect square. Find the square root of the product.
7. How many numbers lie between the squares of the following numbers?
   (i) 16 and 17 (ii) 99 and 100
8. In the following pattern, fill in the missing numbers:

$$1^2 + 2^2 + 2^2 = 3^2$$

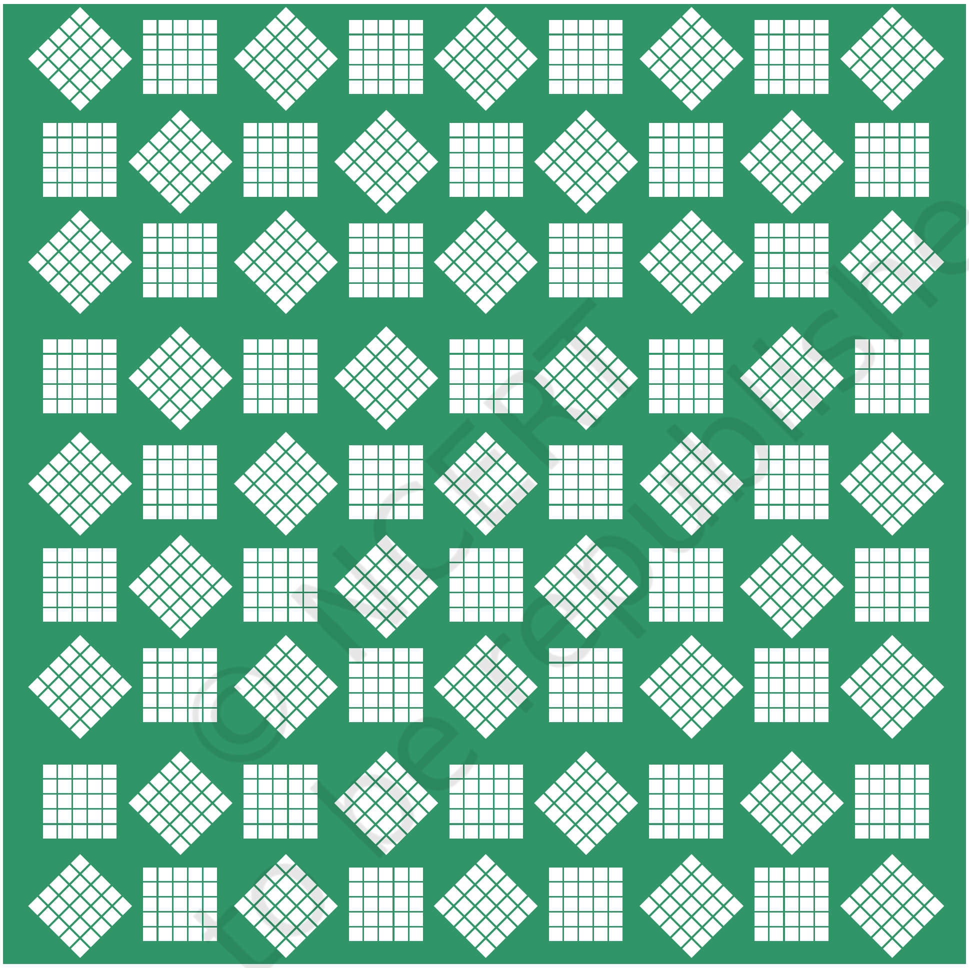
$$2^2 + 3^2 + 6^2 = 7^2$$

$$3^2 + 4^2 + 12^2 = 13^2$$

$$4^2 + 5^2 + 20^2 = (\_\_\_)^2$$

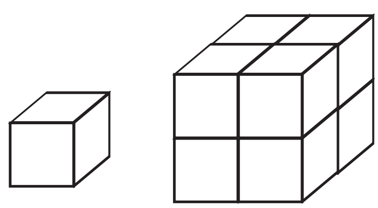
$$9^2 + 10^2 + (\_\_\_)^2 = (\_\_\_)^2$$

9. How many tiny squares are there in the following picture? Write the prime factorisation of the number of tiny squares.



1.2 Cubic Numbers

You know the word **cube** from geometry. A cube is a solid figure all of whose all sides meet at right angles and are equal. How many cubes of side 1 cm make a cube of side 2 cm?



? How many cubes of side 1 cm will make a cube of side 3 cm?

Consider the numbers 1, 8, 27, ...

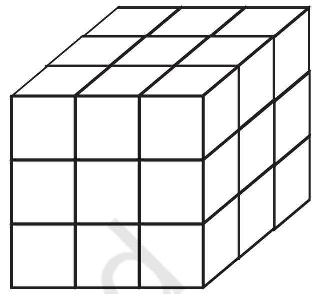
These numbers are called **perfect cubes.** Can you see why they are named so?

Each of them is obtained by multiplying a number by itself three times. We note that

$$1 = 1 \times 1 \times 1$$

$$8 = 2 \times 2 \times 2$$

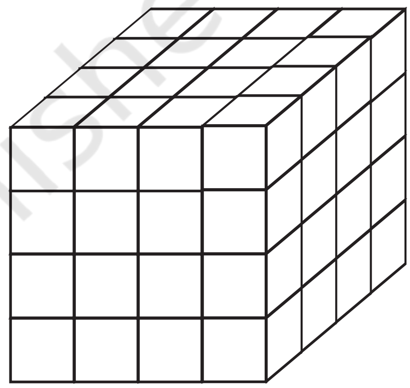
$$27 = 3 \times 3 \times 3$$



? Is 9 a cube?

We see that $2 \times 2 \times 2 = 8$ and $3 \times 3 \times 3 = 27$. This shows that 9 is not a perfect cube. Nor is any number from 10 to 26.

? Can you estimate the number of unit cubes in a cube with an edge length of 4 units?



It has 64 unit cubes! If you notice carefully, each layer of this cube has 4×4 unit cubes. Each square layer has 16 unit cubes (4×4), and there are 4 such layers, so the total number of unit cubes is $4 \times 4 \times 4 = 64$.

Since $5^3 = 5 \times 5 \times 5 = 125$, 125 is a cube.

In general, for any number *n*, we write the cube $n \times n \times n$ as n^3.

? Complete the table below.

$1^3 = 1$	$11^3 = 1331$
$2^3 = 8$	$12^3 =$
$3^3 = 27$	$13^3 = 2197$
$4^3 = 64$	$14^3 = 2744$
$5^3 = 125$	$15^3 =$
$6^3 =$	$16^3 =$
$7^3 =$	$17^3 = 4913$
$8^3 =$	$18^3 = 5832$
$9^3 =$	$19^3 = 6859$
$10^3 =$	$20^3 =$

? What patterns do you notice in the table above?

Math Talk

? We know that 0, 1, 4, 5, 6, 9 are the only last digits possible for squares. What are the possible last digits of cubes?

? Similar to squares, can you find the number of cubes with 1 digit, 2 digits, and 3 digits? What do you observe?

? Can a cube end with exactly two zeroes (00)? Explain.

Just as we can take squares of fractions/decimals — $(\frac{4}{6})^2$, $(13.08)^2$, and $(-6)^2$ — we also can compute cubes of such numbers — $(\frac{4}{6})^3$, $(13.08)^3$, and $(-6)^3$.

$$(\frac{4}{6})^3 = (\frac{4}{6}) \times (\frac{4}{6}) \times (\frac{4}{6}) = (\frac{64}{216})$$

$$(13.08)^3 = 13.08 \times 13.08 \times 13.08 = 2237.810112$$

$$(-6)^3 = -6 \times -6 \times -6 = -216.$$

Taxicab Numbers

Once when Srinivasa Ramanujan was working with G. H. Hardy at the University of Cambridge, Hardy had come to visit Ramanujan at a hospital when he was ill. Hardy had ridden in a taxicab numbered 1729 and he remarked that 1729 was 'rather a dull number,' adding that he hoped that this was not a bad sign. Ramanujan immediately replied, "No, Hardy, it is a very interesting number. It is the smallest number that can be expressed as the sum of two cubes in two different ways".

$$1729 = 1^3 + 12^3$$

$$= 9^3 + 10^3.$$

Because of this story, 1729 has since been known as the **Hardy–Ramanujan** Number. And numbers that can be expressed as the sum of two cubes in two different ways are called **taxicab numbers.**

? The next two taxicab numbers after 1729 are 4104 and 13832. Find the two ways in which each of these can be expressed as the sum of two positive cubes.

How did Ramanujan know this? Well, he loved numbers. All through his life, he tinkered with numbers. During Ramanujan's time in Cambridge, his colleagues often marveled at his ability to see deep patterns in numbers that seemed arbitrary to others. His colleague, John Littlewood, once said, "Every positive integer was one of his [Ramanujan's] personal friends".

Perfect Cubes and Consecutive Odd Numbers

Consecutive odd numbers have a role to play with cubes too. Look at the following pattern:

$$1 = 1 = 1^3$$

$$3 + 5 = 8 = 2^3$$

$$7 + 9 + 11 = 27 = 3^3$$

$$13 + 15 + 17 + 19 = 64 = 4^3$$

$$21 + 23 + 25 + 27 + 29 = 125 = 5^3$$

$$31 + 33 + 35 + 37 + 39 + 41 = 216 = 6^3.$$

Later in this series, we get the following set of consecutive numbers:

$$91 + 93 + 95 + 97 + 99 + 101 + 103 + 105 + 107 + 109.$$

? Can you tell what this sum is without doing the calculation?

Cube Roots

We know that $8 = 2^3$.

We call 2 the cube root of 8 and denote this by $2 = \sqrt[3]{8}$.

More generally, if $y = x^3$, then x is the cube root of y. This is denoted by $x = \sqrt[3]{y}$. So, $\sqrt[3]{8} = \sqrt[3]{2^3} = 2$.

Similarly, $\sqrt[3]{27} = \sqrt[3]{3^3} = 3$ and $\sqrt[3]{1000} = \sqrt[3]{10^3}$ =3. In general, $\sqrt[3]{n^3} = n$.

How do we find out if a number is a cube? Taking inspiration from the case of squares, let us see if we can use prime factorisations.

? Let us check if 3375 is a perfect cube.

$$3375 = 3 \times 3 \times 3 \times 5 \times 5 \times 5.$$

Can the factors be split into three identical groups? For 3375, we can form three groups of (3 × 5). So,

$$3375 = (3 \times 5) \times (3 \times 5) \times (3 \times 5)$$
$$= (3 \times 5)^3 = 15^3.$$

Another way is to check if the factors can be grouped into triplet(s): $3375 = (3 \times 3 \times 3) \times (5 \times 5 \times 5) = 3^3 \times 5^3$.

This means $\sqrt[3]{3375} = 15$.

? Is 500 a perfect cube?

$500 = 2 \times 2 \times 5 \times 5 \times 5$. We see that the factors cannot be split into three identical groups. Therefore, 500 is not a perfect cube.

Prime Factorisation of a Number	Prime Factorisation of its Cube
$4 = 2 \times 2$	$4^3 = 64 = 2 \times 2 \times 2 \times 2 \times 2 \times 2 = 2^3 \times 2^3$
$6 = 2 \times 3$	$6^3 = 216 = 2 \times 2 \times 2 \times 3 \times 3 \times 3 = 2^3 \times 3^3$
$15 = 3 \times 5$	$15^3 = 3375 = 3 \times 3 \times 3 \times 5 \times 5 \times 5 = 3^3 \times 5^3$
$12 = 2 \times 2 \times 3$	$12^3 = 1728 = 2 \times 2 \times 2 \times 2 \times 2 \times 2 \times 3 \times 3 \times 3 = 2^3 \times 2^3 \times 3^3$

Observe that each prime factor of a number appears three times in the prime factorisation of its cube.

? Find the cube roots of these numbers:

(i) $\sqrt[3]{64}$ = (ii) $\sqrt[3]{512}$ = (iii) $\sqrt[3]{729}$ =

Successive Differences

We know that the differences between consecutive perfect squares gives the sequence of odd numbers. Observe the figure below where the differences are computed successively for perfect squares. After two levels, all the differences are the same.

Perfect Squares

1 4 9 16 25 36 ...

Level 1 **3 5 7 9 11 ...**

Level 2 **2 2 2 2 ...**

? Compute successive differences over levels for perfect cubes until all the differences at a level are the same. What do you notice?

Perfect Cubes

1 8 27 64 125 216 ...

1.3 A Pinch of History

The first known list of perfect squares and perfect cubes was compiled by the Babylonians as far back as 1700 BCE. These lists, found on clay tablets, were used to quickly find square roots and cube roots in problems involving land measurement, architectural design, and other areas where geometric calculations were necessary.

In ancient Sanskrit works the term *varga* was used both for the square figure or its area, as well as the square power, and the term *ghana* was used both for the solid cube as well as the product of a number with itself three times. The fourth power was called *varga-varga* . These terms were used in India at least from the third century BCE.

Aryabhata (499 CE) states

"A square figure of four equal sides and the number representing its area are called *varga*. The product of two equal quantities is also called *varga*."

Thus, the term *varga* for square power has its origin in the graphical representation of a square figure.

Why is the word 'root' (the root of a plant) used for the mathematical operation √ (square root, cube root, etc.)?

It is because, in ancient India, the Sanskrit word *mula*, meaning root of a plant, basis, cause, origin, etc., was used for the mathematical operations of taking roots.

In Sanskrit, *varga-mula* (the basis, cause, origin of the square) was used for square-root and *ghana-mula* was used for cube-root. This use of *mula* for the mathematical concept of root was subsequently emulated in Arabic and Latin through their corresponding words for the root of a plant — *jidhr* and *radix* respectively. The term *mula* for root has been used in India at least from the first century BCE. Another term used was *pada* (foot, basis, cause, origin). Brahmagupta (628 CE) explains, 'The *pada* (root) of a *krti* (square) is that of which it is a square.'

Figure it Out

1. Find the cube roots of 27000 and 10648.
2. What number will you multiply by 1323 to make it a cube number?
3. State true or false. Explain your reasoning.
   (i) The cube of any odd number is even.
   (ii) There is no perfect cube that ends with 8.
   (iii) The cube of a 2-digit number may be a 3-digit number.
   (iv) The cube of a 2-digit number may have seven or more digits.
   (v) Cube numbers have an odd number of factors.
4. You are told that 1331 is a perfect cube. Can you guess without factorisation what its cube root is? Similarly, guess the cube roots of 4913, 12167, and 32768.

5. Which of the following is the greatest? Explain your reasoning.
 (i) $67^3 - 66^3$ (ii) $43^3 - 42^3$ (iii) $67^2 - 66^2$ (iv) $43^2 - 42^2$

SUMMARY

- A number obtained by multiplying a number by itself is called a **square number**. Squares of natural numbers are called **perfect squares**.
- All perfect squares end with 0, 1, 4, 5, 6 or 9. Squares can only have an even number of zeros at the end.
- **Square root** is the inverse operation of square. Every perfect square has two integral square roots. The positive square root of a number is denoted by the symbol $\sqrt{}$. For example, $\sqrt{9} = 3$.
- A **number** obtained by multiplying a number by itself three times is called a **cube**. For example 1, 8, 27, ... ,etc., are cubes.
- A number is a perfect square if its prime factors can be split into two identical groups.
- A number is a perfect cube if its prime factors can be split into three identical groups
- The symbol $\sqrt[3]{}$ denotes cube root. For example, $\sqrt[3]{27} = 3$.

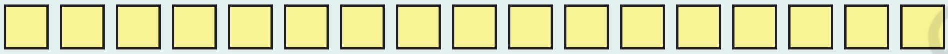
Square Pairs!

Look at the following numbers: 3 6 10 15 1
They are arranged such that each pair of adjacent numbers adds up to a square.

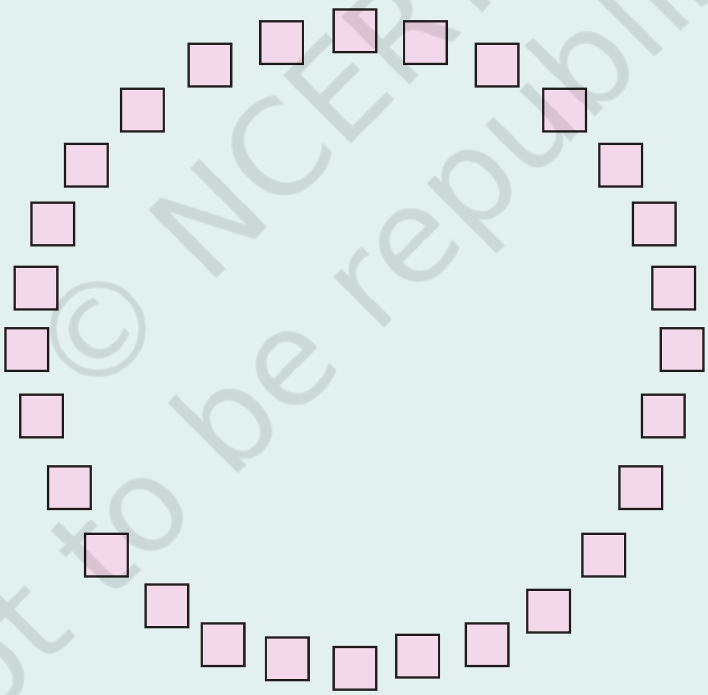
3 + 6 = 9, 6 + 10 = 16, 10 + 15 = 25, 15 + 1 = 16.

Try arranging the numbers 1 to 17 (without repetition) in a row in a similar way — the sum of every adjacent pair of numbers should be a square.

Can you arrange them in more than one way? If not, can you explain why?



Can you do the same with numbers from 1 to 32 (again, without repetition), but this time arranging all the numbers in a circle?



2 POWER PLAY

2.1 Experiencing the Power Play ...

An Impossible Venture!

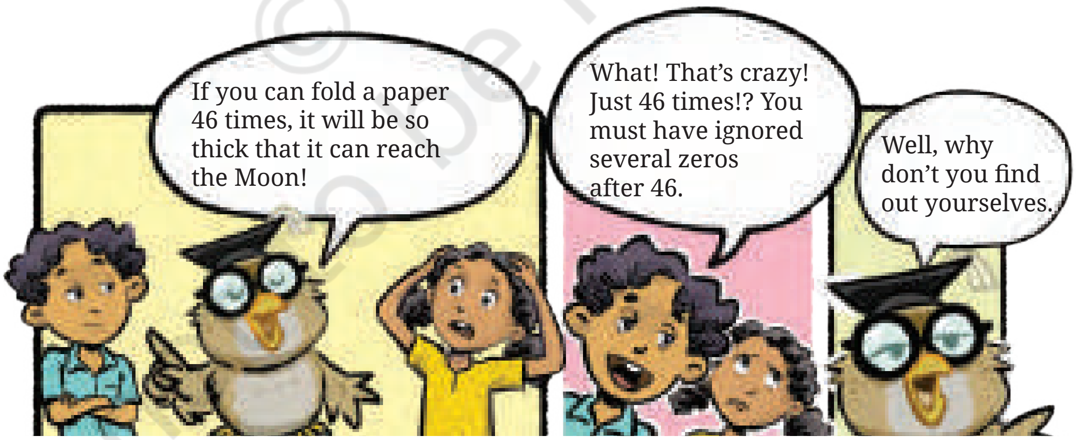
Take a sheet of paper, as large a sheet as you can find. Fold it once. Fold it again, and again.

? How many times can you fold it over and over?

Estu says "I heard that a sheet of paper can't be folded more than 7 times".

Roxie replies "What if we use a thinner paper, like a newspaper or a tissue paper?"

Try it with different types of paper and see what happens.



? Say you can fold a sheet of paper as many times as you wish. What would its thickness be after 30 folds? Make a guess.

Let us find out how thick a sheet of paper will be after 46 folds. Assume that the thickness of the sheet is 0.001 cm.

The following table lists the thickness after each fold. Observe that the thickness doubles after each fold.

Fold	Thickness	Fold	Thickness	Fold	Thickness
1	0.002 cm	7	0.128 cm	13	8.192 cm
2	0.004 cm	8	0.256 cm	14	16.384 cm
3	0.008 cm	9	0.512 cm	15	32.768 cm
4	0.016 cm	10	1.024 cm	16	65.536 cm
5	0.032 cm	11	2.048 cm	17	≈ 131 cm
6	0.064 cm	12	4.096 cm		

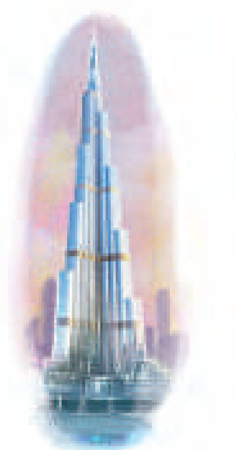
(We use the sign '≈' to indicate 'approximately equal to'.)
After 10 folds, the thickness is just above 1 cm (1.024 cm).
After 17 folds, the thickness is about 131 cm (a little more than 4 feet).

Now, what do you think the thickness would be after 30 folds? 45 folds? Make a guess. Math Talk

Fill the table below.

Fold	Thickness	Fold	Thickness	Fold	Thickness
18	≈ 262 cm	21		24	
19	≈ 524 cm	22		25	
20	≈ 10.4 m	23		26	

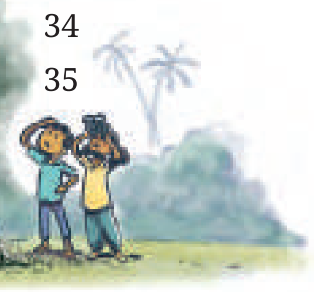
After 26 folds, the thickness is approximately 670 m. Burj Khalifa in Dubai, the tallest building in the world, is 830 m tall.



Fold	Thickness	Fold	Thickness
27	≈ 1.3 km	29	
28		30	

After 30 folds, the thickness of the paper is about 10.7 km, the typical height at which planes fly. The deepest point discovered in the oceans is the Mariana Trench, with a depth of 11 km.

Fold	Thickness	Fold	Thickness	Fold	Thickness
31		36		41	
32		37		42	
33		38		43	
34		39		44	
35		40		45	



It might be hard to digest the fact that after just 46 folds, the thickness is more than 7,00,000 km. This is the power of **multiplicative growth,** also called **exponential growth.** Let us analyse the growth.
We have seen that the thickness doubles after every fold.

Fold 4	0.016 cm
Fold 5	0.032 cm

Fold 9	0.512 cm
Fold 10	1.024 cm

Notice the change in thickness after two folds. By how much does it increase?

Fold 4	0.016 cm
Fold 6	0.064 cm

After any 3 folds, the thickness increases 8 times (= 2 × 2 × 2). Check if that is true. Similarly, from any point, the thickness after 10 folds increases by 1024 times (= 2 multiplied by itself 10 times), as shown in the table below.

Fold	Thickness	Times increased by
0 to 10	1.024 cm – 0.001 cm = 1.023 cm	1.024 ÷ 0.001 = 1024
10 to 20	10.485 m – 1.024 cm ≈ 10.474 m	10.485 m ÷ 1.024 cm = 1024
20 to 30	10.737 km – 10.485 m ≈ 10.726 km	10.737 km ÷ 10.485 m = 1024
30 to 40	10995 km – 10.737 km ≈ 10984.2 km	10995 km ÷ 10.737 km = 1024

2.2 Exponential Notation and Operations

The initial thickness of the paper was 0.001 cm.
Upon folding once, its thickness became 0.001 cm × 2 = 0.002 cm.

Folding it twice, its thickness became —
0.001 cm × 2 × 2 = 0.004 cm, or 0.001 cm × 2^2 = 0.004 cm (in shorthand).

Upon folding it thrice, its thickness became —
0.001 cm × 2 × 2 × 2, or 0.001 cm × 2^3 = 0.008 cm.

When folded four times, its thickness became —
0.001 cm × 2 × 2 × 2 × 2, or 0.001 cm × 2^4 = 0.016 cm.

Similarly, the expression for the thickness of the paper when folded 7 times will be 0.001 cm × 2 × 2 × 2 × 2 × 2 × 2 × 2, or 0.001 cm × 2^7 = 0.128 cm.

We have seen that square numbers can be expressed as n^2 and cube numbers as n^3.

$n \times n = n^2$ (read as '*n* squared' or '*n* raised to the power 2')

$n \times n \times n = n^3$ (read as 'n cubed' or 'n raised to the power 3')
$n \times n \times n \times n = n^4$ (read as 'n raised to the power 4' or 'the 4th power of n')
$n \times n \times n \times n \times n \times n \times n = n^7$ (read as 'n raised to the power 7' or 'the 7th power of n') and so on.

In general, we write n^a to denote n multiplied by itself a times.

$$5^4 = 5 \times 5 \times 5 \times 5 = 625.$$

5^4 is the exponential form of 625. Here, 4 is the **exponent/power**, and 5 is the **base.** Exponents of the form 5^n are called powers of 5: 5^1, 5^2, 5^3, 5^4, etc. $2 \times 2 \times 2 \times 2 \times 2 \times 2 \times 2 \times 2 \times 2 \times 2 = 2^{10} = 1024$. Remember the 1024 from earlier? There, it meant that after every 10 folds, the thickness increased 1024 times.

5^4 is read as
'5 raised to the power 4' or
'5 to the power 4' or
'5 power 4' or
'4th power of 5'

? Which expression describes the thickness of a sheet of paper after it is folded 10 times? The initial thickness is represented by the letter-number v.

(i) $10v$	(ii) $10 + v$	(iii) $2 \times 10 \times v$
(iv) 2^{10}	(v) $2^{10}v$	(vi) 10^2v

Some more examples of exponential notation:
$4 \times 4 \times 4 = 4^3 = 64$.
$(-4) \times (-4) \times (-4) = (-4)^3 = -64$.
Similarly,

$a \times a \times a \times b \times b$ can be expressed as a^3b^2 (read as a cubed b squared).
$a \times a \times b \times b \times b \times b$ can be expressed as a^2b^4 (read as a squared b raised to the power 4).
Remember that $4 + 4 + 4 = 3 \times 4 = 12$, whereas $4 \times 4 \times 4 = 4^3 = 64$.

? Express the number 32400 as a product of its prime factors and represent the prime factors in their exponential form.

2	32400
2	16200
2	8100
2	4050
5	2025
5	405
3	81
3	27
3	9
3	3
	1

$$32400 = 2 \times 2 \times 2 \times 2 \times 5 \times 5 \times 3 \times 3 \times 3 \times 3.$$

In exponential form, this would be

$$32400 = 2^4 \times 5^2 \times 3^4.$$

? What is $(-1)^5$? Is it positive or negative? What about $(-1)^{56}$?

? Is $(-2)^4 = 16$? Verify.

What is 0^2, 0^5?
What is 0^n?

? Figure it Out

1. Express the following in exponential form:

(i) $6 \times 6 \times 6 \times 6$	(ii) $y \times y$
(iii) $b \times b \times b \times b$	(iv) $5 \times 5 \times 7 \times 7 \times 7$
(v) $2 \times 2 \times a \times a$	(vi) $a \times a \times a \times c \times c \times c \times c \times d$

2. Express each of the following as a product of powers of their prime factors in exponential form.

   (i) 648 (ii) 405 (iii) 540 (iv) 3600

3. Write the numerical value of each of the following:

   (i) 2×10^3 (ii) $7^2 \times 2^3$ (iii) 3×4^4
   (iv) $(-3)^2 \times (-5)^2$ (v) $3^2 \times 10^4$ (vi) $(-2)^5 \times (-10)^6$

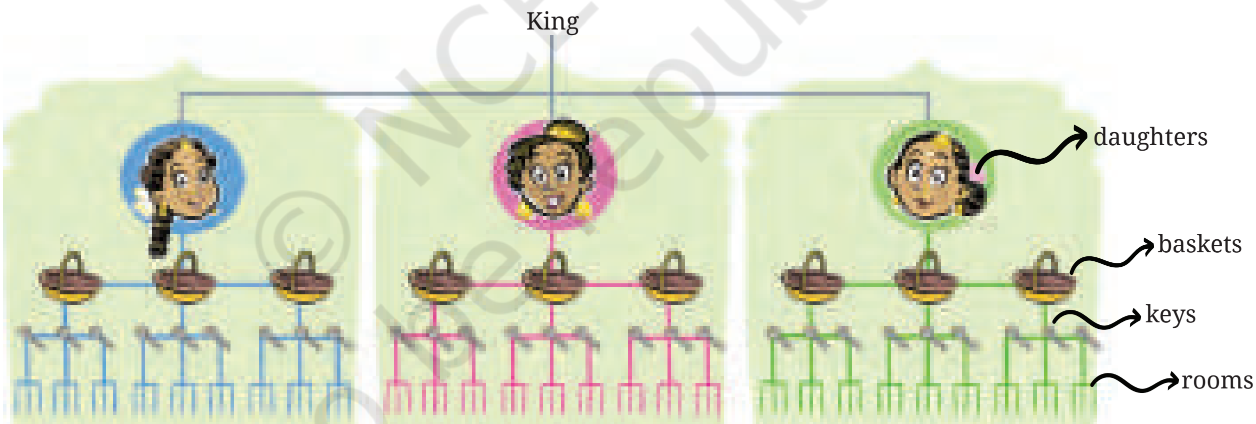
The Stones that Shine ...

? Three daughters with curious eyes,
Each got three baskets — a kingly prize.
Each basket had three silver keys,
Each opens three big rooms with ease.
Each room had tables — one, two, three,
With three bright necklaces on each, you see.
Each necklace had three diamonds so fine...
Can you count these stones that shine?

Hint: Find out the number of baskets and rooms.

? How many rooms were there altogether?

The information given can be visualised as shown below.



From the diagram, the number of rooms is 3^4. This can be computed by repeatedly multiplying 3 by itself,

$$3 \times 3 = 9.$$
$$9 \times 3 = 27.$$
$$27 \times 3 = 81.$$
$$81 \times 3 = 243.$$

? How many diamonds were there in total? Can we find out by just one multiplication using the products above?

The number of diamonds is $3 \times 3 \times 3 \times 3 \times 3 \times 3 \times 3 = 3^7$.

We can write

$$3^7 = (3 \times 3 \times 3 \times 3) \times (3 \times 3 \times 3)$$

We had computed till 3^4. To find 3^7, we can just multiply 3^4 (= 81) with 3^3 (= 27).

$$= 3^4 \times 3^3$$
$$= 81 \times 27 = 2187$$

$$\underbrace{3 \times 3 \times 3 \times 3}_{3^4} \times \underbrace{3 \times 3 \times 3}_{3^3}$$

? 3^7 can also be written as $3^2 \times 3^5$. Can you reason out why?

This can be easily extended to products where exponents are the same letter-numbers.

? Write the product $p^4 \times p^6$ in exponential form.

$$p^4 \times p^6 = (p \times p \times p \times p) \times (p \times p \times p \times p \times p \times p) = p^{10}.$$

We can generalise this to —

$n^a \times n^b = n^{a+b}$, where a and b are counting numbers.

? Use this observation to compute the following.

(i) 2^9 (ii) 5^7 (iii) 4^6

4^6 can be evaluated in these two ways,

$(4 \times 4 \times 4) \times (4 \times 4 \times 4) = 4^3 \times 4^3$ $= 64 \times 64$ $= 4096.$ $4^3 \times 4^3$ is the square of 4^3, i.e., $4^3 \times 4^3$ can also be written as $(4^3)^2$.	$(4 \times 4) \times (4 \times 4) \times (4 \times 4) = 4^2 \times 4^2 \times 4^2$ $= 16 \times 16 \times 16$ $= 4096.$ $4^2 \times 4^2 \times 4^2$ is the cube of 4^2, i.e., $4^2 \times 4^2 \times 4^2$ can also be written as $(4^2)^3$.

Similarly, $7^4 = (7 \times 7) \times (7 \times 7) = 7^2 \times 7^2 = (7^2)^2$, and

$2^{10} = (2 \times 2) \times (2 \times 2) \times (2 \times 2) \times (2 \times 2) \times (2 \times 2)$
$= (2^2) \times (2^2) \times (2^2) \times (2^2) \times (2^2)$
$= (2^2)^5.$

? Is 2^{10} also equal to $(2^5)^2$? Write it as a product.

$2^{10} = (2 \times 2 \times 2 \times 2 \times 2) \times (2 \times 2 \times 2 \times 2 \times 2)$
$= (2^5) \times (2^5)$
$= (2^5)^2.$

In general,

$(n^a)^b = (n^b)^a = n^{a \times b} = n^{ab}$, where a and b are counting numbers.

? Write the following expressions as a power of a power in at least two different ways:

(i) 8^6 (ii) 7^{15} (iii) 9^{14} (iv) 5^8

Magical Pond

? In the middle of a beautiful, magical pond lies a bright pink lotus. The number of lotuses doubles every day in this pond. After 30 days, the pond is completely covered with lotuses. On which day was the pond half full?

If the pond is completely covered by lotuses on the 30th day, how much of it is covered by lotuses on the 29th day?

Since the number of lotuses doubles every day, the pond should be half covered on the 29th day.

? Write the number of lotuses (in exponential form) when the pond was —

(i) fully covered (ii) half covered

? There is another pond in which the number of lotuses triples every day. When both the ponds had no flowers, Damayanti placed a lotus in the doubling pond. After 4 days, she took all the lotuses from there and put them in the tripling pond. How many lotuses will be in the tripling pond after 4 more days?

After the first 4 days, the number of lotuses is $1 \times 2 \times 2 \times 2 \times 2 = 2^4$.
After the next 4 days, the number of lotuses is $2^4 \times 3 \times 3 \times 3 \times 3 = 2^4 \times 3^4$.

? What if Damayanti had changed the order in which she placed the flowers in the lakes? How many lotuses would be there?

$$1 \times 3^4 \times 2^4 = (3 \times 3 \times 3 \times 3) \times (2 \times 2 \times 2 \times 2).$$

? Can this product be expressed as an exponent m^n, where m and n are some counting numbers?

By regrouping the numbers,

$$= (3 \times 2) \times (3 \times 2) \times (3 \times 2) \times (3 \times 2)$$
$$= (3 \times 2)^4 = 6^4.$$

In general form,

$$m^a \times n^a = (mn)^a \text{, where } a \text{ is a counting number.}$$

Use this observation to compute the value of $2^5 \times 5^5$.

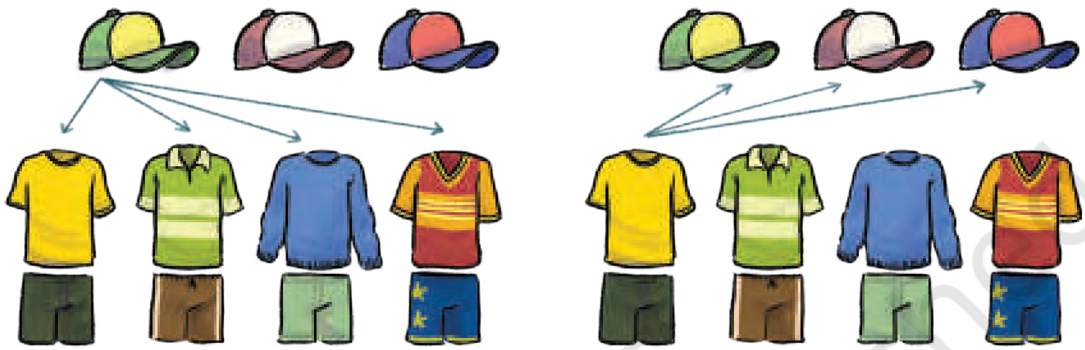
? Simplify $\frac{10^4}{5^4}$ and write it in exponential form.

In general, we can show that $\frac{m^a}{n^a} = \left(\frac{m}{n}\right)^a$.

How Many Combinations

? Estu has 4 dresses and 3 caps. How many different ways can Estu combine the dresses and caps?

For each cap, he can choose any of the 4 dresses, so for 3 caps, 4 + 4 + 4 = 4 × 3 = 12 combinations are possible. We can also look at it as—for each dress, Estu can choose any of the 3 caps, so for 4 outfits, 3 + 3 + 3 + 3 = 3 × 4 = 12 combinations are possible.



? Roxie has 7 dresses, 2 hats, and 3 pairs of shoes. How many different ways can Roxie dress up?

Hint: Try drawing a diagram like the one above.

? Estu and Roxie came across a safe containing old stamps and coins that their great-grandfather had collected. It was secured with a 5-digit password. Since nobody knew the password, they had no option except to try every password until it opened. They were unlucky and the lock only opened with the last password, after they had tried all possible combinations. How many passwords did they end up checking?

If you can't solve a problem, try to find a simpler version of the problem that you can solve. This technique can come in handy many times.

Instead of a 5-digit lock, let us assume we have a 2-digit lock and try to find out how many passwords are possible.

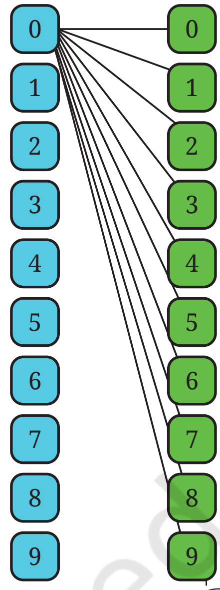
There are 10 options for the first digit (0 to 9). For each of these, there are 10 options for the second digit (If 0 is the first digit then 00, 01, 02, 03, ..., 09 are possible). Therefore the total number of combinations for a 2-digit lock is 10 × 10 = 100.

Now, suppose we have a 3-digit lock. For each of the earlier 100 (2-digit) passwords there are 10 choices for the third digit. So, there are 100 × 10 = 1000 combinations for a 3-digit lock. You can list them all: 000, 001, 002, ..., 997, 997, 999.

How many 5-digit passwords are possible?

Each digit has 10 choices, so a 5-digit lock will have:

$10 \times 10 \times 10 \times 10 \times 10 = 10^5 = 1{,}00{,}000$ passwords. This is the same as writing numbers till 99,999 with all 5 digits, i.e., 00000, 00001, 00002, ...00010, 00011, ..., 00100, 00101, ..., 00999, ..., 30456, ..., 99998, 99999.



Estu says, "Next time, I will buy a lock that has 6 slots with the letters A to Z. I feel it is safer."

? How many passwords are possible with such a lock?

? Think about how many combinations are possible in different contexts. Some examples are—

Try This

(i) Pincodes of places in India—The Pincode of Vidisha in Madhya Pradesh is 464001. The Pincode of Zemabawk in Mizoram is 796017.

(ii) Mobile numbers.

(iii) Vehicle registration numbers.

Try to find out how these numbers or codes are allotted/generated.

2.3 The Other Side of Powers

Imagine a line of length 16 units. Erasing half of it would result in

$$2^4 \div 2 = \frac{2 \times 2 \times 2 \times 2}{2} = 2 \times 2 \times 2 = 2^3 = 8 \text{ units.}$$

Erasing half one more time would result in,

$$(2^4 \div 2) \div 2 = 2^4 \div 2^2 = \frac{2 \times 2 \times 2 \times 2}{2 \times 2} = 2 \times 2 = 2^2 = 4 \text{ units.}$$

Halving 16 cm three times may be written as,

$$2^4 \div 2^3 = \frac{2 \times 2 \times 2 \times 2}{2 \times 2 \times 2} = 2 = 2^1 = 2 \text{ units.}$$

From this we can see that

$$2^4 \div 2^3 = 2^{4-3} = 2^1.$$

? What is $2^{100} \div 2^{25}$ in powers of 2?

In a generalised form,

$$n^a \div n^b = n^{a-b},$$

where $n \neq 0$ and a and b are counting numbers and $a > b$.

Why can't n be 0?

Math Talk

We have not covered the case when the exponent is 0; for example, what is 2^0?

Let us define 2^0 in a way that the generalised form above holds true.

$$2^0 = 2^{4-4} = 2^4 \div 2^4 = \frac{2 \times 2 \times 2 \times 2}{2 \times 2 \times 2 \times 2} = 1.$$

In fact for any letter number a

$$2^0 = 2^{a-a} = 2^a \div 2^a = 1.$$

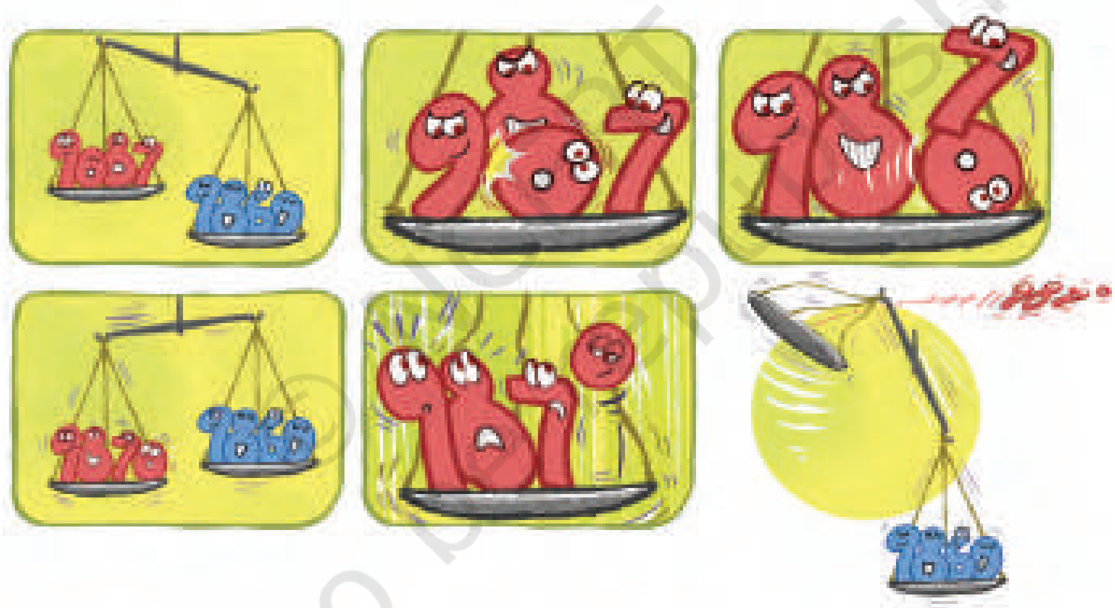
In general,

$$x^a \div x^a = x^{a-a} = x^0, \text{ and so}$$

$$1 = x^0,$$

where $x \neq 0$ and a is a counting number.

When Zero is in Power!



When a line of length 2^4 units is halved 5 times,

$$2^4 \div 2^5 = \frac{2 \times 2 \times 2 \times 2}{2 \times 2 \times 2 \times 2 \times 2} = \frac{1}{2} \text{ units.}$$

Using the generalised form, we get $2^4 \div 2^5 = 2^{(4-5)} = 2^{-1}$.

So, $2^{-1} = \frac{1}{2}$.

When a line of length 2^4 units is halved 10 times, we get $2^4 \div 2^{10} = 2^{(4-10)} = 2^{-6}$ units.

When expanded, $2^4 \div 2^{10} = \frac{2 \times 2 \times 2 \times 2}{2 \times 2 \times 2 \times 2 \times 2 \times 2 \times 2 \times 2 \times 2 \times 2} = \frac{1}{2^6} = \frac{1}{64}$, which is also written as 2^{-6}.

Similarly, $10^{-3} = \frac{1}{10^3}$, $7^{-2} = \frac{1}{7^2}$, etc.

? Can we write $10^3 = \frac{1}{10^{-3}}$?

We can write,

$$\frac{1}{10^{-3}} = \frac{1}{1/10^3} = 1 \div \frac{1}{10^3} = 1 \times 10^3 = 10^3.$$

Similarly, $7^2 = \frac{1}{7^{-2}}$ and $4^a = \frac{1}{4^{-a}}$.

In a generalised form,

$$n^{-a} = \frac{1}{n^a} \text{ and } n^a = \frac{1}{n^{-a}}\text{, where } n \neq 0.$$

Math Talk

Consider the following general forms we have identified.

$n^a \times n^b = n^{a+b}$	$(n^a)^b = (n^b)^a = n^{a \times b}$	$n^a \div n^b = n^{a-b}$

? We had required a and b to be counting numbers. Can a and b be any integers? Will the generalised forms still hold true?

? Write equivalent forms of the following.

(i) 2^{-4} (ii) 10^{-5} (iii) $(-7)^{-2}$

(iv) $(-5)^{-3}$ (v) 10^{-100}

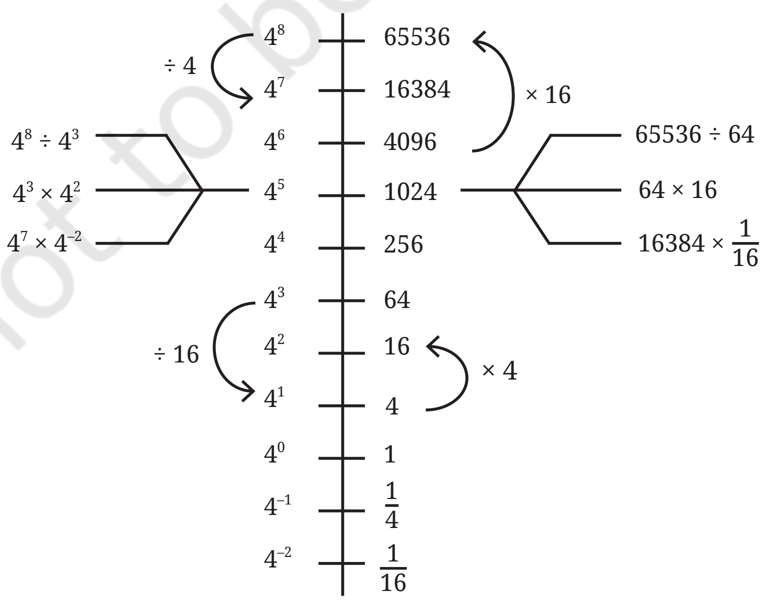
? Simplify and write the answers in exponential form.

(i) $2^{-4} \times 2^7$ (ii) $3^2 \times 3^{-5} \times 3^6$ (iii) $p^3 \times p^{-10}$

(iv) $2^4 \times (-4)^{-2}$ (v) $8^p \times 8^q$

Power Lines

Let us arrange the powers of 4 along a line.



? Can we say that 16384 (4^7) is 16 (4^2) times larger than 1,024 (4^5)?
Yes, since $4^7 \div 4^5 = 4^2$.

? How many times larger than 4^{-2} is 4^2?

? Use the power line for 7 to answer the following questions.

7^7	823543
7^6	117649
7^5	16807
7^4	2401
7^3	343
7^2	49
7^1	7
7^0	1
7^{-1}	$\frac{1}{7}$
7^{-2}	$\frac{1}{49}$
7^{-3}	$\frac{1}{343}$
7^{-4}	$\frac{1}{2401}$

$2{,}401 \times 49 =$

$49^3 =$

$343 \times 2{,}401 =$

$\frac{16{,}807}{49} =$

$\frac{7}{343} =$

$\frac{16{,}807}{8{,}23{,}543} =$

$1{,}17{,}649 \times \frac{1}{343} =$

$\frac{1}{343} \times \frac{1}{343} =$

2.4 Powers of 10

We have used numbers like 10, 100, 1000, and so on when writing Indian numerals in an expanded form. For example,

$$47561 = (4 \times 10000) + (7 \times 1000) + (5 \times 100) + (6 \times 10) + 1.$$

This can be written using powers of 10 as

$$(4 \times 10^4) + (7 \times 10^3) + (5 \times 10^2) + (6 \times 10^1) + (1 \times 10^0).$$

? Write these numbers in the same way: (i) 172, (ii) 5642, (iii) 6374.

? How can we write 561.903?

$$561.903 = (5 \times 100) + (6 \times 10) + 1 + (9 \times \frac{1}{10}) + (0 \times \frac{1}{100}) + (3 \times \frac{1}{1000}).$$

Writing it using powers of 10, we have

$$561.903 = (5 \times 10^2) + (6 \times 10^1) + (1 \times 10^0) + (9 \times 10^{-1}) + (0 \times 10^{-2}) + (3 \times 10^{-3}).$$

Scientific Notation

Let's look at some facts involving large numbers—

(i) The Sun is located 30,00,00,00,00,00,00,00,00,00,000 m from the centre of our Milky Way galaxy.

(ii) The number of stars in our galaxy is 1,00,00,00,00,000.

(iii) The mass of the Earth is 59,76,00,00,00,00,00,00,00,00,00,000 kg.

As the number of digits increases, it becomes difficult to read the numbers correctly. We may miscount the number of zeroes or place commas incorrectly. We will then read the wrong value. It is like getting ₹5,000 when you were supposed to get ₹50,000. The number of zeroes is more important than the initial digits in several cases.

Can we use the exponential notation to simplify and read these very large numbers correctly?

For example, the number 5900 can be expressed as—

$$5900 = 590 \times 10 = 590 \times 10^1$$
$$= 59 \times 100 = 59 \times 10^2$$
$$= 5.9 \times 1000 = 5.9 \times 10^3$$
$$= 0.59 \times 10000 = 0.59 \times 10^4.$$

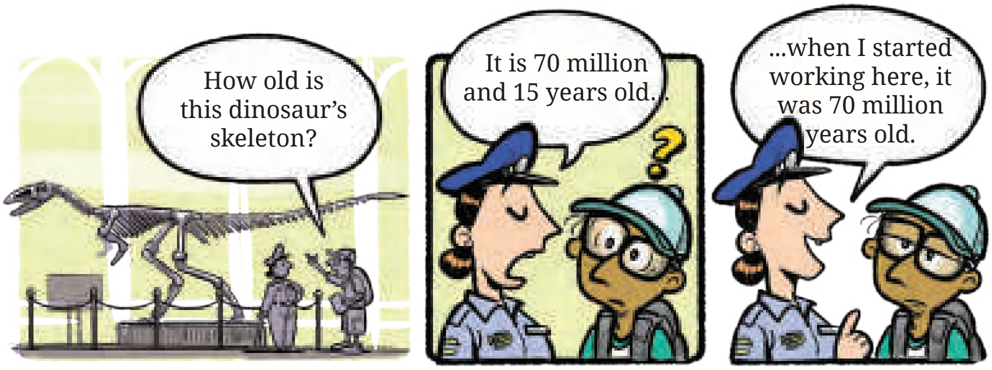
Any number can be written as the product of a number between 1 and 10 and a power of 10. For example,

$5900 = 5.9 \times 10^3$ $\quad$ $20800 = 2.08 \times 10^4$ $\quad$ $80,00,000 = 8 \times 10^6$

? Write the large-number facts we read just before in this form.

In **scientific notation or scientific form** (also called **standard form)**, we write numbers as $x \times 10^y$, where $x \geq 1$ and $x < 10$ is the coefficient and y, the exponent, is any integer. Often, the exponent y is more important than the coefficient x. When we write the 2 crore population of Mumbai as 2×10^7, the 7 is more important than the 2. Indeed, if the 2 is changed to 3, the population increases by one-half, i.e., 2 crore to 3 crore, whereas if the 7 is changed to 8, the change in population is 10 times, i.e., 2 crores to 20 crores. Therefore, the standard form explicitly mentions the exponent, which indicates the number of digits.

If we say that the population of Kohima is 1,42,395, then it gives the impression that we are quite sure about this number up to the units place. When we use large numbers, in most cases, we are more concerned about how big a quantity or measure is, rather than the exact value. If we are only sure that the population is around 1 lakh 42 thousand, we can write it as 1.42×10^5. If we can only be certain that it is around 1 lakh 40 thousand, we write it as 1.4×10^5. The number of digits in the coefficient reflects how well we know the number. The most important part of any number written in scientific form is the exponent, and then the first digit of the coefficient. The digits following the coefficient are small corrections to the first digit.



These values are rounded-off estimates, averages, or approximations; most of the time, they serve the purpose at hand.

The distance between the Sun and Saturn is 14,33,50,00,00,000 m = 1.4335×10^{12} m.

The distance between Saturn and Uranus is 14,39,00,00,00,000 m = 1.439×10^{12} m. The distance between the Sun and Earth is 1,49,60,00,00,000 m = 1.496×10^{11} m.

? Can you say which of the three distances is the smallest?

? The number line below shows the distance between the Sun and Saturn (1.4335×10^{12} m). On the number line below, mark the relative position of the Earth. The distance between the Sun and the Earth is 1.496×10^{11} m.

? Express the following numbers in standard form.

(i) 59,853 (ii) 65,950

(iii) 34,30,000 (iv) 70,04,00,00,000

2.5 Did You Ever Wonder?

Last year, we looked at interesting thought experiments in the chapter on Large Numbers. Let us continue this journey.

Nanjundappa wants to donate jaggery equal to Roxie's weight and wheat equal to Estu's weight. He is wondering how much it would cost.

? What would be the worth (in rupees) of the donated jaggery? What would be the worth (in rupees) of the donated wheat?

In order to find out, let us first describe the relationships among the quantities present.

Worth of jaggery (in rupees) = Roxie's weight in kg × cost of 1 kg jaggery.

Worth of wheat (in rupees) = Estu's weight in kg × cost of 1 kg wheat.

? Make necessary and reasonable assumptions for the unknowns and find the answers. Remember, Roxie is 13 years old and Estu is 11 years old.

Assuming Roxie's weight to be 45 kg and the cost of 1 kg of jaggery to be ₹70, the worth of donated jaggery is 45 × 70 = ₹3150. Assuming Estu's weight to be 50 kg and the cost of 1 kg of wheat to be ₹50, the worth of donated wheat is 50 × 50 = ₹2500.

The practice of offering goods equal to the weight of a person, called Tulābhāra or Tulābhāram, is quite old and is still followed in many places in Southern India. It is a symbol of bhakti (surrendering oneself), a token of gratitude; it also supports the community.

? Roxie wonders, "Instead of jaggery if we use 1-rupee coins, how many coins are needed to equal my weight?". How can we find out?

For questions like these, you can consider following the steps suggested below.

1. Guessing: Make an instinctive (quick) guess of what the answer could be, without any calculations.

2. Calculating using estimation and approximation —
   (i) Describe the relationships among the quantities that are needed to find the answer.
   (ii) Make reasonable assumptions and approximations if the required information is not available.
   (iii) Compute and find the answer (and check how close your guess was).

? Would the number of coins be in hundreds, thousands, lakhs, crores, or even more? Make an instinctive guess.

? Find the answer by making necessary and reasonable assumptions and approximations for the unknowns. Remember, we are not looking for an exact answer but a reasonably close estimate.

How about measuring to find out the weight of a 1-rupee coin?

Initially, your guesses may be very far off from the answer and it is perfectly fine! You will get better at it like as you do it often and in different situations. Guessing and estimating can build intuition about numbers and various quantities.

Estu asks, "What if we use 5-rupee coins or 10-rupee notes instead? How much money could it be?"

? Make an instinctive guess first. Then find out (make necessary and reasonable assumptions about the unknown details and find the answers).

Estu says, "When I become an adult, I would like to donate notebooks worth my weight every year". Roxie says, "When I grow up, I would like to do *annadāna* (offering grains or meals) worth my weight every year".

? How many people might benefit from each of these offerings in a year? Again, guess first before finding out.

Math Talk

Roxie and Estu overheard someone saying—"We did *pādayātra* for about 400 km to reach this place! We arrived early this morning."

? How long ago would they have started their journey?

? Find answers by making necessary assumptions and approximations. Do guess first before calculating to check how close your guess was!

Note to the Teacher: Assumptions can vary greatly at times, and as a result the answers computed using these you can also vary. This is perfectly alright. Modelling the situation properly is crucial, which can also be done in different ways sometimes. The accuracy of the assumed numbers or quantities can get better with exposure and practice.

Pādayātra, is the traditional practice of walking long distances as part of a religious or spiritual pursuit. People across religions in our country observe similar forms of pilgrimage or spiritual walking, although they may have different names or purposes.

Some of the pilgrimages are Ajmer Sharif Dargah Ziyarat, Pandharpur Wari, Kānwar Yatra, Sabarimala Yatra, Sammed Shikharji Yatra, Lumbini to Sarnath Yatra.

Before the rise of modern transport, people moved from one place to another by walking — sometimes merchants, sages, and scholars walked thousands of kilometres to different parts of the world across deserts, mountains, and rivers.

? How many times can a person circumnavigate (go around the world) the Earth in their lifetime if they walk non-stop? Consider the distance around the Earth as 40,000 km.

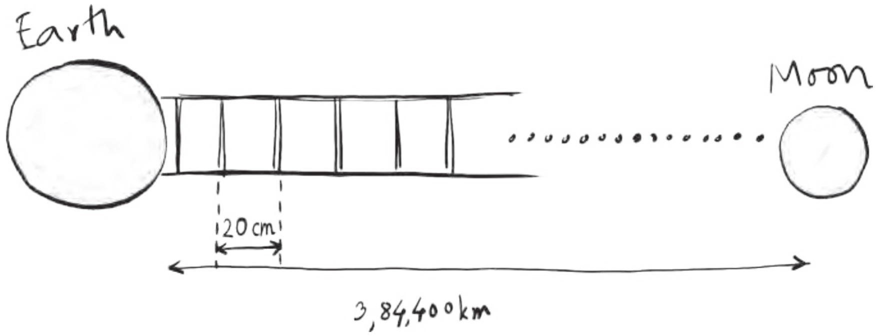
Linear Growth vs. Exponential Growth

Roxie tells Estu about a science-fiction novel she is reading where they build a ladder to reach the moon, "... I wonder if we actually had a ladder like that, how many steps would it have?".

? What do you think? Make an instinctive guess first.

? Would the number of steps be in thousands, lakhs, crores, or even more?

To find out, we would need to know the gap between consecutive steps of the ladder. Let's assume a reasonable distance of 20 cm. Visualising the problem as shown,



? We have to find out how many 20 cm make 3,84,400 km.

If we calculate the value, we get the result as 1,92,20,00,000 steps, which is 192 crore and 20 lakh steps or 1 billion 922 million steps. The fixed increase in the distance from the earth with each step (a 20 cm gain after each step) is called **linear growth.**

To cover the distance between the Earth and the Moon, it takes 1,92,20,00,000 steps with linear growth whereas it takes just 46 folds of a piece of paper with exponential growth! Linear growth is additive, whereas exponential growth is multiplicative.

$$\underbrace{20 + 20 + 20 + \ldots\ldots\ldots\ldots}_{\text{1,92,20,00,000 times}} \qquad \underbrace{0.001 \times 2 \times 2 \times 2 \ldots\ldots\ldots\ldots}_{\text{46 times}}$$

Some examples of exponential growth we have seen earlier in this chapter are 'The Stones that Shine', 'Magical Pond', 'How Many Combinations'. We shall explore more such interesting examples in a later chapter and also in the next grade.

? Can you come up with some examples of linear growth and of exponential growth?

Getting a Sense for Large Numbers

Last year, we learnt about lakhs and crores, as well as millions and billions. A lakh is 10^5 (1,00,000), a crore is 10^7 (1,00,00,000), and an arab is 10^9 (1,00,00,00,000), whereas a million is 10^6 (1,000,000) and a billion is 10^9 (1,000,000,000).

You might know the size of the world's human population. Have you ever wondered how many ants there might be in the world or how long ago humans emerged? In this section, we shall explore numbers significantly larger than arabs and billions. We shall use powers of 10 to represent and compare these numbers in each case.

10^0 As of mid-2025, there are only two northern white rhinos remaining in the world, both females, and they reside at the Ol Pejeta Conservancy in Kenya (= 2×10^0).

10^1 As of early 2024, the total population of Hainan gibbons is a meagre 42 ($\approx 4 \times 10^1$).

10^2 There are just 242 Kakapo alive as of mid-2025 ($\approx 2 \times 10^2$).

10^3 There are fewer than 3000 Komodo dragons in the world, all based in Indonesia ($\approx 3 \times 10^3$).

10^4 A 2005 estimate of the maned wolf population showed that there are more than 17000 of them; most are located in Brazil (1.7×10^4).

10^5 As of 2018, there are around 4.15 lakh African elephants ($\approx 4 \times 10^5$).

10^6 There are an estimated 50 lakh / 5 million American alligators as of 2025 (5×10^6).

10^7 The global camel population is estimated to be over 3.5 crore/ 35 million (3.5×10^7). India has only about 2.5 lakhs of them. The global horse population is around 5.8 crore / 58 million (5.8×10^7), with about half of them in America.

10^8 More than 20 crore / 200 million (2×10^8) water buffaloes are estimated worldwide, with a vast majority of them in Asia.

10^9 The estimated global population of starlings is around 1.3 arab/1.3 billion (_________). The global human population as of 2025 is 8.2 arab/8.2 billion (8.2×10^9).

A picture of a starling murmuration over a farm in the UK. Starling murmuration is a mesmerising aerial display of thousands of starlings flying in synchronised, swirling patterns. It is often described as a 'choreographed dance'.

? With a global human population of about 8×10^9 and about 4×10^5 African elephants, can we say that there are nearly 20,000 people for every African elephant?

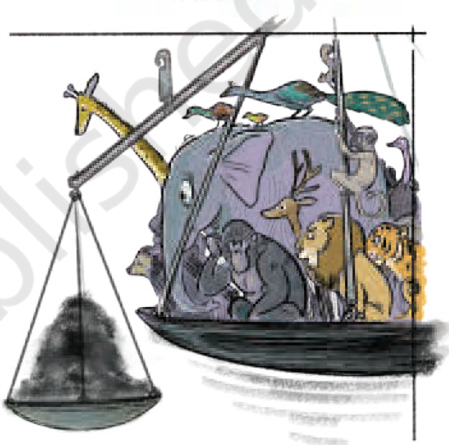
10^{10} The global chicken population living at any time is estimated at ≈ 33 billion (3.3×10^{10}).

10^{12} The estimated number of trees (2023) globally stands at 30 kharab/3 trillion (3×10^{12}). One kharab is 100 arab, and one trillion is 1000 billion.

10^{14} The estimated mosquito population worldwide (2023) is 11 neel/110 trillion (________). A derived estimate of the population of the Antarctic krill stands at 50 neel/500 trillion (5×10^{14}).

10^{15} An estimate of the beetle population stands at 1 padma/1 quadrillion (1×10^{15}). The estimate of the earthworm population is also at 1 padma/1 quadrillion.

10^{16} The estimated population of ants globally is 20 padma/20 quadrillion (2×10^{16}). Ants alone outweigh all wild birds and wild mammals combined.



10^{21} is supposed to be the number of grains of sand on all beaches and deserts on Earth. This is enough sand to give every ant 10 little sand castles to live in.

10^{23} The estimated number of stars in the observable universe is 2×10^{23}.

10^{25} There are an estimated 2×10^{25} drops of water on Earth (assuming 16 drops per millilitre).

? Calculate and write the answer using scientific notation:

(i) How many ants are there for every human in the world?

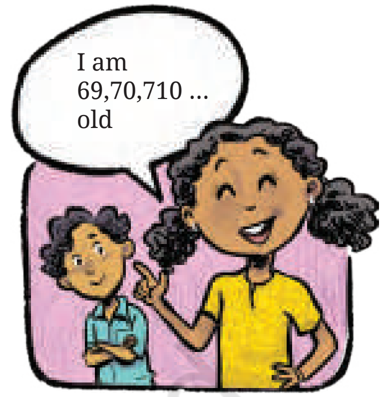
(ii) If a flock of starlings contains 10,000 birds, how many flocks could there be in the world?

(iii) If each tree had about 10^4 leaves, find the total number of leaves on all the trees in the world.

(iv) If you stacked sheets of paper on top of each other, how many would you need to reach the Moon?

A different way to say your age!

"How old are you?" asked Estu.
"I completed 13 years a few weeks ago!" said Roxie.
"How old are you?" asked Estu again.
"I'm 4840 days old today!" said Roxie.
"How old are you?" asked Estu again.
"I'm ______ hours old!" said Roxie.
Make an estimate before finding this number.
Estu: "I am 4070 days old today. Can you find out my date of birth?"



What could this number mean? Find out!

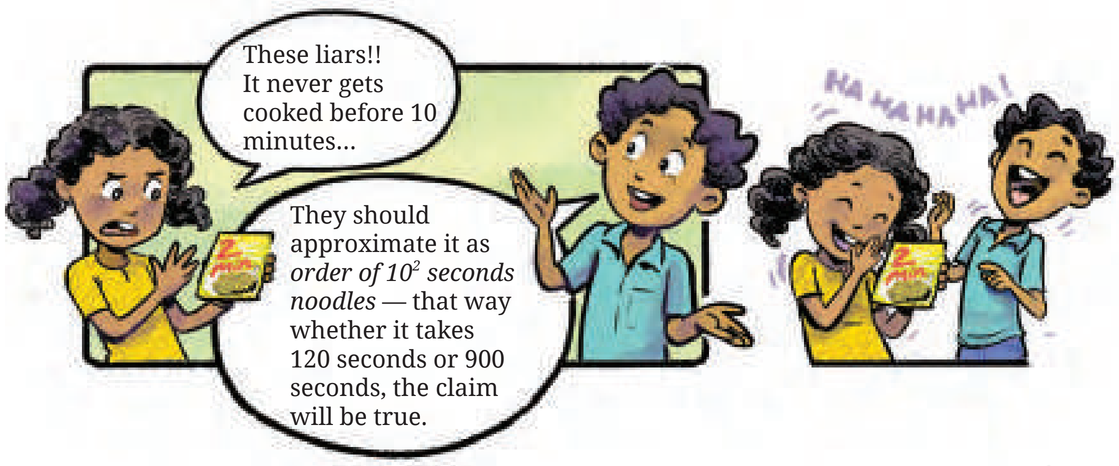
? If you have lived for a million seconds, how old would you be?

We shall look at approximate times and timelines of some events and phenomena, and use powers of 10 to represent and compare these quantities.

Time in seconds	Comparison to real-world events/phenomena
10^0 = 1 second	- Time taken for a ball thrown up to fall back on the ground (typically a few seconds).
10^1 = 10 seconds	- Time blood takes to complete one full circulation through the body: 10–20 seconds ($1 \times 10^1 - 2 \times 10^1$ seconds).
	- Typical waiting time at a traffic signal.

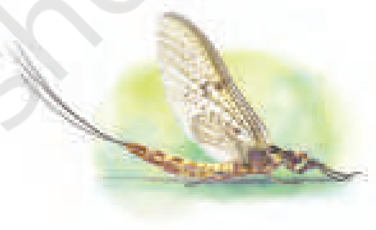
Isn't it quite amazing how someone is able to estimate things like the number of ants in the world or the time blood takes to fully circulate? You may carry this wonder whenever you encounter such facts. You will come across such facts in subjects like Science and Social Science, where such estimates are made frequently.

10^2 seconds ≈1.6 minutes	- Time needed to make a cup of tea: 5–10 minutes ($\approx 4 \times 10^2 - 8 \times 10^2$ seconds).
	- Time for light to reach the Earth from the Sun: about 8 minutes ($\approx 5 \times 10^2$ seconds).



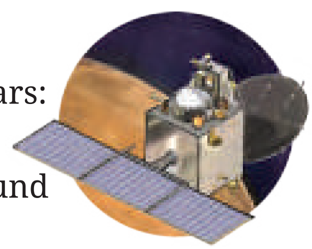
10^3 seconds ≈ 16.6 minutes
- Satellites in low Earth orbits take between 90 minutes (≈ 5.5×10^3 seconds) to 2 hours to complete one full revolution around the Earth.

10^4 seconds ≈ 2.7 hours
- The time needed to digest a meal: about 2–4 hours to pass through the stomach.
- Lifespan of an adult mayfly: about a day (≈ 9×10^4 seconds).



? 10^5 seconds ≈ 1.16 days and 10^6 seconds ≈ 11.57 days. Think of some events or phenomena whose time is of the order of (i) 10^5 seconds and (ii) 10^6 seconds. Write them in scientific notation.

10^7 seconds ≈ 115.7 days / ≈ 3.8 months
- Time spent sleeping in a year: about 4 months.
- Time taken by Mangalyaan mission to reach Mars: 298 days (≈ 2.65×10^7 seconds).
- Time taken by Mars for one full revolution around the Sun: 687 Earth-days/1.88 Earth-years (≈ 6×10^7 seconds).

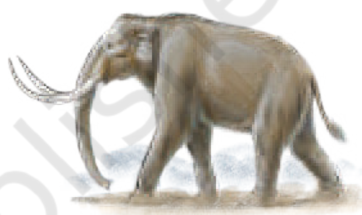


10^8 seconds ≈ 3.17 years
- The typical lifespan of most dogs is 3 to 15 years.

10^9 seconds ≈ 31.7 years
- The orbital period of Halley's comet is 75–79 years; the next expected return is in the year 2061 (≈ 2.4×10^9 seconds).
- Duration of one full revolution of Neptune around the Sun: 60,190 Earth-days/~165 Earth-years or 89,666 Neptunian days/1 Neptunian-year (≈ 5.2×10^9 seconds). A day on Neptune is about 16.1 hours

Notice how rapid exponential growth is — 10^6 seconds is less than a fortnight, but 10^9 seconds is a whopping 31 years (about half the life expectancy of a human)!

10^{10} seconds ≈ 317 years	-	The Chola dynasty ruled for more than 900 years (≈ 3×10^{10} seconds) between the 3rd Century BCE and 12th Century CE.
10^{11} seconds ≈ 3,170 years	-	Age of the oldest known living tree: about 5000 years (≈ 1.57×10^{11} seconds).
	-	Time since the last peak ice age: 19,000 – 26,000 years ago (≈ 6×10^{11} seconds – 8.2×10^{11} seconds).
10^{12} seconds ≈ 31,700 years	-	Early Homo sapiens first appeared 2–3 lakh years ago (≈7×10^{12} – 9×10^{12} seconds). The entire population around that time could fit in a large cricket stadium.
10^{13} seconds ≈ 3.17 lakh years	-	The Steppe Mammoth is estimated to have appeared around 8–18 lakh years ago.



10^{14} seconds ≈ 3.17 million years	-	A fossil of Kelenken Guillermoi, a type of terror bird, is dated to 15 million years ago (≈______________ seconds).

10^{15} seconds ≈ 3.17 crore years	-	Age of Himalayas: 5.5 crore years/55 million years (≈ 1.7 × 10^{15} seconds); they continue to grow a few mm every year.
	-	Dinosaurs went extinct 6.6 crore years ago/66 million years ago (≈ 2×10^{15} seconds).
	-	Dinosaurs first appeared more than 20 crore/200 million years ago (≈ 6×10^{15} seconds).
	-	It takes about 23 crore years for the Sun to make one complete trip around the Milky Way (≈ 7×10^{15} seconds).

10^{16} seconds ≈ 31.7 crore years
- Plants on land started 47 crore/470 million years ago (≈ ______________ seconds).

10^{17} seconds ≈ 3.17 billion years
- The oldest fossil evidence suggests that bacteria first appeared about 3.7 billion years ago.
- The Earth is 4.5 billion years old.
- The Milky Way galaxy was formed 13.6 billion years ago, and the Universe was formed 13.8 billion years ago.

Notice that 10^9 seconds is of the order of the lifespan of a human, whereas 10^{18} seconds ago the universe did not exist according to modern physics!! The exponential notation can capture very large quantities in a concise manner.

? Calculate and write the answer using scientific notation:

(i) If one star is counted every second, how long would it take to count all the stars in the universe? Answer in terms of the number of seconds using scientific notation.

(ii) If one could drink a glass of water (200 ml) every 10 seconds, how long would it take to finish the entire volume of water on Earth?

Try This

Very large quantities are often beyond our experience and comprehension. To put them into perspective, we can relate and compare them with quantities we are familiar with. This can give an essence of how large a number or a measure is!

2.5 A Pinch of History

In the *Lalitavistara*, a Buddhist treatise from the first century BCE, we see number-names for odd powers of ten up to 10^{53}. The following occurs as part of the dialogue between the mathematician Arjuna and Prince Gautama, the *Bodhisattva*.

"Hundred *kotis* are called an *ayuta* (10^9), hundred *ayutas* a *niyuta* (10^{11}), hundred *niyutas* a *kankara* (10^{13}), …, hundred *sarva-balas* a *visamjna-gati* (10^{47}), hundred *visamjna-gatis* a *sarvajna* (10^{49}), hundred *sarvajnas* a *vibhutangama* (10^{51}), a hundred *vibhutangamas* is a *tallakshana* (10^{53})."

Mahaviracharya gives a list of 24 terms (i.e., up to 10^{23}) in his treatise *Ganita-sara-sangraha*. An anonymous Jaina treatise *Amalasiddhi* gives a list with a name for each power of ten up to 10^{96} *(dasha-ananta)*. A Pali grammar treatise of *Kāccāyana* lists number-names up to 10^{140}, named *asaṅkhyeya*.

For expressing high powers of ten, Jaina and Buddhist texts use bases like *sahassa* (thousand) and *koṭi* (ten million); for instance, *prayuta* (10^6) would be *dasa sata sahassa* (ten hundred thousand).

The modern naming is similar to this, where we say,

A hundred thousand is a lakh	$100 \times 1000 = 1{,}00{,}000$	$10^2 \times 10^3 = 10^5$
A hundred lakhs is a crore	$100 \times 1{,}00{,}000 = 1{,}00{,}00{,}000$	$10^2 \times 10^5 = 10^7$
A hundred crores is an arab	$100 \times 1{,}00{,}00{,}000 = 1{,}00{,}00{,}00{,}000$	$10^2 \times 10^7 = 10^9$
A hundred arab is a kharab	$100 \times 1{,}00{,}00{,}00{,}000$ $= 1{,}00{,}00{,}00{,}00{,}000$	$10^2 \times 10^9 = 10^{11}$

Continuing this, a hundred kharab is a neel (10^{13}), a hundred *neel* is a padma (10^{15}), a hundred *padma* is a *shankh* (10^{17}) and a hundred *shankh* is a *maha shankh* (10^{19}).

In the American/International system, we say

A thousand thousand is a million	$1000 \times 1000 = 1{,}000{,}000$	$10^3 \times 10^3 = 10^6$
A thousand million is a billion	$1000 \times 1{,}000{,}000 = 1{,}000{,}000{,}000$	$10^3 \times 10^6 = 10^9$
A thousand billion is an trillion	$1000 \times 1{,}000{,}000{,}000$ $= 1{,}00{,}000{,}000{,}000$	$10^3 \times 10^9 = 10^{12}$

Continuing this, a thousand trillion is a quadrillion (10^{15}). This pattern continues. Observe the names **mil**lion (10^6), **bi**llion (10^9), **tri**llion (10^{12}), **quad**rillion (10^{15}), **quin**tillion (10^{18}), **sext**illion (10^{21}), **sept**illion (10^{24}), **oct**illion (10^{27}), **non**illion (10^{30}), **dec**illion (10^{33}).

? What does the first part of each name denote?

The number 10^{100} is also called a googol. The estimated number of atoms in the universe is 10^{78} to 10^{82}. The number 10^{googol} is called a googolplex. It is hard to imagine how large this number is!

The currency note with the highest denomination in India currently is 2000 rupees. Guess what is the highest denomination of a currency note ever, across the world. The highest numerical value banknote ever printed was a special note valued 1 sextillion pengő (10^{21} or 1 milliard bilpengő) printed in Hungary in 1946, but it was never issued. In 2009, Zimbabwe printed a 100 trillion (10^{14}) Zimbabwean dollar note, which at the time of printing was worth about $30.

Figure it Out

1. Find out the units digit in the value of $2^{224} \div 4^{32}$? [Hint: $4 = 2^2$]

2. There are 5 bottles in a container. Every day, a new container is brought in. How many bottles would be there after 40 days?

3. Write the given number as the product of two or more powers in three different ways. The powers can be any integers.

   (i) 64^3 (ii) 192^8 (iii) 32^{-5}

4. Examine each statement below and find out if it is 'Always True', 'Only Sometimes True', or 'Never True'. Explain your reasoning.

   (i) Cube numbers are also square numbers.
   (ii) Fourth powers are also square numbers.
   (iii) The fifth power of a number is divisible by the cube of that number.
   (iv) The product of two cube numbers is a cube number.
   (v) q^{46} is both a 4th power and a 6th power (q is a prime number).

5. Simplify and write these in the exponential form.

   (i) $10^{-2} \times 10^{-5}$ (ii) $5^7 \div 5^4$
   (iii) $9^{-7} \div 9^4$ (iv) $(13^{-2})^{-3}$
   (v) $m^5n^{12}(mn)^9$

6. If $12^2 = 144$ what is

   (i) $(1.2)^2$ (ii) $(0.12)^2$
   (iii) $(0.012)^2$ (iv) 120^2

7. Circle the numbers that are the same—

   $2^4 \times 3^6$ $\quad$ $6^4 \times 3^2$ $\quad$ 6^{10} $\quad$ $18^2 \times 6^2$ $\quad$ 6^{24}

8. Identify the greater number in each of the following—

   (i) 4^3 or 3^4 $\quad$ (ii) 2^8 or 8^2 $\quad$ (iii) 100^2 or 2^{100}

9. A dairy plans to produce 8.5 billion packets of milk in a year. They want a unique ID (identifier) code for each packet. If they choose to use the digits 0–9, how many digits should the code consist of?

10. 64 is a square number (8^2) and a cube number (4^3). Are there other numbers that are both squares and cubes? Is there a way to describe such numbers in general?

    Math Talk

11. A digital locker has an alphanumeric (it can have both digits and letters) passcode of length 5. Some example codes are G89P0, 38098, BRJKW, and 003AZ. How many such codes are possible?

12. The worldwide population of sheep (2024) is about 10^9, and that of goats is also about the same. What is the total population of sheep and goats?

    (ii) 20^9 $\quad$ (ii) 10^{11} $\quad$ (iii) 10^{10}

    (iv) 10^{18} $\quad$ (v) 2×10^9 $\quad$ (vi) $10^9 + 10^9$

13. Calculate and write the answer in scientific notation:

    (i) If each person in the world had 30 pieces of clothing, find the total number of pieces of clothing.

    (ii) There are about 100 million bee colonies in the world. Find the number of honeybees if each colony has about 50,000 bees.

    (iii) The human body has about 38 trillion bacterial cells. Find the bacterial population residing in all humans in the world.

    (iv) Total time spent eating in a lifetime in seconds.

14. What was the date 1 arab/1 billion seconds ago?

    Try This

SUMMARY

- We analysed some situations, asked questions, and found answers by first guessing, then modelling the problem statement, followed by making assumptions and approximations to carry out the calculations.
- We experienced how rapid exponential growth, also called multiplicative growth, can be compared to additive growth.
- n^a is $n \times n \times n \times n \times ... \times n$ (n multiplied by itself a times) and $n^{-a} = \frac{1}{n^a}$.
- Operations with exponents satisfy
  - $n^a \times n^b = n^{a+b}$
  - $(n^a)^b = (n^b)^a = n^{a \times b}$
  - $n^a \div n^b = n^{a-b}$ $(n \neq 0)$
  - $n^a \times m^a = (n \times m)^a$
  - $n^a \div m^a = (n \div m)^a$ $(m \neq 0)$
  - $n^0 = 1$ $(n \neq 0)$
- The scientific notation for the number 308100000 is 3.081×10^8. The **standard form** of the **scientific notation** of any number is $x \times 10^y$, where $x \geq 1$ and $x < 10$, and y is an integer.
- Engaging in interesting thought experiments can be used as means to understand how large a number or a quantity is.

It's PUZZLE TIME!

Tremendous in Ten!

Find a partner to play this game with. In 10 seconds, the person who writes a number or an expression, using only the digits 0-9 and arithmetic operations, that gives a number that is the larger between the two wins the round.

10000000000000 $\qquad$ 999999×999999

In Round 1, Roxie wrote 10000000000000 and Estu wrote 999999 × 999999. Between these two, Roxie's number is greater. Can you see why? Roxie's number is 10^{13}, whereas Estu's number is less than $(10^6)^2$.

In Round 2, Roxie wrote $10^{1000} + 10^{1000} + 10^{1000} + 10^{1000}$ and Estu wrote $(10^{1000000}) \times 9000$. Can you say which is greater?

$10^{1000} + 10^{1000} + 10^{1000} + 10^{1000}$ $\qquad$ $10^{1000000} \times 9000$

Below are some conditions that you may consider for different rounds.

(i) Exponents are not allowed. Only addition is allowed.

(ii) Exponents are not allowed. Only addition and multiplication are allowed.

(iii) Exponents are allowed. Only addition is allowed.

(iv) Exponents are allowed. Any arithmetic operation is allowed.

You can create your own conditions and/or involve more people to play together.

3

A STORY OF NUMBERS

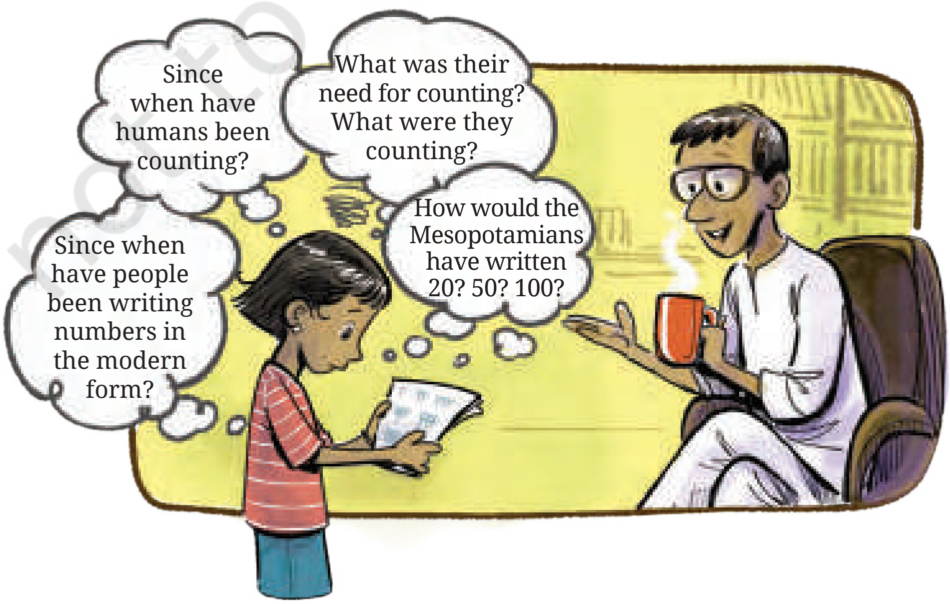
0874CH03

3.1 Reema's Curiosity

One lazy afternoon, Reema was flipping through an old book when—whoosh!—a piece of paper slipped out and floated to the floor. She picked it up and stared at the strange symbols all over it. "What is this?" she wondered.

She ran to her father, holding the paper as if it were a secret treasure. He looked at it and smiled. "Around 4000 years ago, there flourished a civilisation in a region called Mesopotamia, in the western part of Asia, containing a major part of the present-day Iraq and a few other neighbouring countries. This is one of the ways they wrote their numbers!"

Reema's eyes lit up, "Seriously? These strange symbols were numbers?" Her curiosity was sparked, and questions started swirling in her head.



Sensing her curiosity, her father started telling her how the idea of number and number representation evolved over the course of time, across geographies, to finally reach its modern efficient form.
Get ready to travel back in time with them!

Humans had the need to count even as early as the Stone Age. They were counting to determine the quantity of food they had, the number of animals in their livestock, details regarding trades of goods, the number of offerings given in rituals, etc. They also wanted to keep track of the passing days, e.g., to know and predict when important events such as the new moon, full moon, or onset of a season would occur. However, when they said or wrote down such numbers, they didn't make use of the numbers that we use today.

The structure of the modern oral and written numbers that we use today had its origin thousands of years ago in India. Ancient Indian texts, such as the *Yajurveda Samhita*, mentioned names of numbers based on powers of 10, almost as we say them orally today. For example, they listed names for the numbers one *(eka)*, ten *(dasha)*, hundred *(shata)*, thousand (*sahasra*), ten thousand *(āyuta)*, etc., all the way up to 10^{12} and beyond.

The way we write our numbers today — using the digits 0 through 9 — also originated and were developed in India, around 2000 years ago. The first known instance of numbers being written using ten digits, including the digit 0 (which was then notated as a dot), occurs in the *Bakhshali* manuscript (c. 3rd century CE). Aryabhata (c. 499 CE) was the first mathematician to fully explain, and do elaborate scientific computations with the Indian system of 10 symbols.

Zero in the *Bakhshali* manuscript

The Indian number system was transmitted to the Arab world by around 800 CE. It was popularised in the Arab world by the great Persian mathematician Al-Khwārizmī (after whom the word 'algorithm' is named) through his book *On the Calculation with Hindu Numerals* (*c.* 825) and by the noted philosopher Al-Kindi through his work *On the Use of the Hindu Numerals* (*c.* 830).

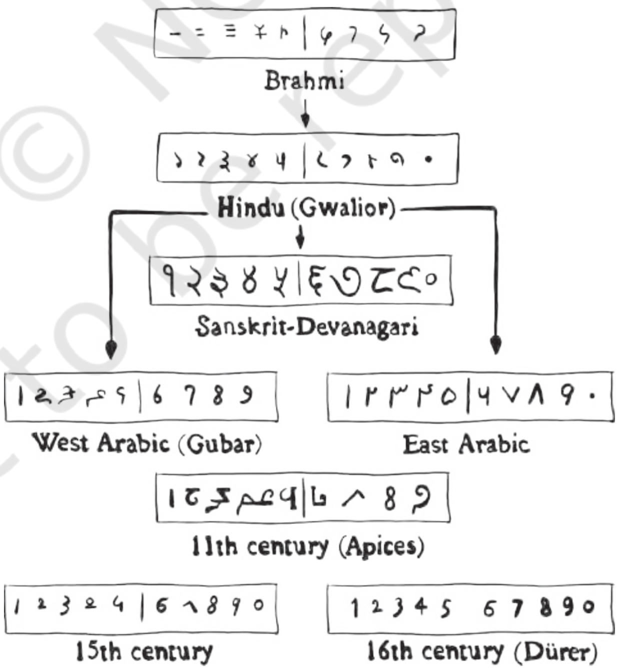
From the Arab world, the Hindu numerals were transmitted to Europe and to parts of Africa by around 1100 CE. Though Al-Khwārizmī's work on calculation with Hindu numerals was translated into Latin, it was the Italian mathematician Fibonacci who around the year 1200 really made the case to Europe to adopt the Indian numerals. However, the Roman numerals were so ingrained in European thinking and writing at the time that the Indian numerals did not gain widespread use for several more centuries. But eventually, during the European Renaissance and by the 17th century, not adopting them became impossible or it would impede scientific progress.

> *"The ingenious method of expressing every possible number using a set of ten symbols (each symbol having a place value and an absolute value) emerged in India. The idea seems so simple nowadays that its significance and profound importance is no longer appreciated. Its simplicity lies in the way it facilitated calculations and placed arithmetic foremost among useful inventions."*
>
> ***— Pierre-Simon Laplace (1749 – 1827)***

Their use then spread to every continent, and are now used in every corner of the world.

Because European scholars learned the Indian numerals from the Arab world, they called them 'Arabic numerals' to reflect their European perspective. On the other hand, as noted above, Arab scholars, such as Al-Khwārizmī and Al-Kindi, called them 'Hindu numerals'. During the period of European colonisation, the European term Arabic numbers became widely used. However, in recent years, this mistake is being corrected in many textbooks and documents around the world, including in Europe. The most commonly used terminologies for the numbers we use today are 'Hindu numerals', 'Indian numerals', and the transitional 'Hindu-Arabic numerals'. It is worth noting that the word 'Hindu' here does not refer to a religion, but rather a geography/people from whom these numbers came.

The shape of the digits 0, 1, 2, ..., 9 used to write numbers in the Indian number system today evolved over a period of time, as shown below:



Evolution of the digits used in the Indian number system

Prior to the global adoption of the Indian system of numerals, different groups of people used different methods of representing numbers. We shall take a glimpse of some of them. We will not be looking at different systems in a chronological order, but rather an order that shows us the main stages in the development of the idea of number representation.

But first, let us explore some of the foundational ideas needed to count and to determine the number of objects in a given collection.

The Mechanism of Counting

Imagine that we are living in the Stone Age, say, around ten thousand years ago. Suppose we have a herd of cows. Here are some natural questions that we might ask about our herd—

Q1. How do we ensure that all cows have returned safely after grazing?

Q2. Do we have fewer cows than our neighbour?

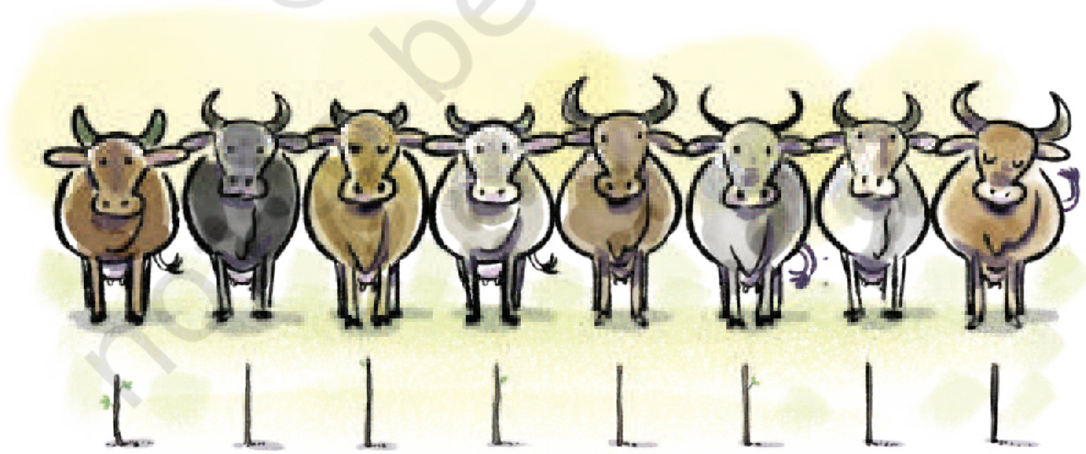
Q3. If there are fewer, how many more cows would we need so that we have the same number of cows as our neighbour?

Math Talk

We need to tackle these questions without the use of the number names or written numbers of the Hindu number system. How do we do it?

Here are some possible methods.

Method 1: We could tackle the questions by using pebbles, sticks or any object that is available in abundance. Let us choose sticks. For every cow in the herd, we could keep a stick. The final collection of sticks tells us the number of cows, which can be used to check if any cows have gone missing.



This way of associating each cow with a stick, such that no two cows are associated or mapped to the same stick is called a one-to-one mapping. This mapping can then be used to come up with a way to represent numbers, as shown in the table.

Number	Its representation (using sticks)
1	|
2	| |
3	| | |
4	| | | |
5	| | | | |
.	.
.	.
.	.

? How will you use such sticks to answer the other two questions (Q2 and Q3)?

Method 2: Instead of objects, we could use a standard sequence of sounds or names. For example, we could use the sounds of the letters of any language. While counting, we could make a one-to-one mapping between the objects and the letters: that is, associate each object to be counted with a letter, following the letter-order. This mapping can then be used to come up with a way of verbally representing numbers.

For example, we get the following number representation if we use English letters '*a*' to '*z*'.

Number	Its representation (using sounds or names)
1	*a*
2	*b*
3	*c*
4	*d*
5	*e*
.	.
.	.
.	.
26	*z*

An obvious limitation of using only the letters of the English alphabet in this form is that it cannot be used to count collections having more than 26 objects.

? How many numbers can you represent in this way using the sounds of the letters of your language?

Math Talk

Method 3: We could use a sequence of written symbols as follows.

Table 1

Number	1	2	3	4	5	6	7	8	9	10
Representation using symbols	I	II	III	IV	V	VI	VII	VIII	IX	X
Number	11	12	13	14	15	16	17	18	19	20
Representation using symbols	XI	XII	XIII	XIV	XV	XVI	XVII	XVIII	XIX	XX

? Do you see a way of extending this method to represent bigger numbers as well? How?

From the discussion above, we see that for counting and finding the size of a collection, we need a standard sequence of objects, or names, or written symbols, that has a fixed order. Let us call this standard sequence a **number system.** A collection of objects can be counted by making a one-to-one mapping between them and the standard sequence, following the sequence order.

Since there is no end to numbers, the challenge is to come up with an unending standard sequence/number system that is easy to count with. Using sticks gives an unending standard sequence/number system. However, it is not convenient to count larger collections, as we will need as many sticks as the number of objects being counted. Using the sounds of the letters of a language, as in Method 2, is convenient for the counting process but is not an unending standard sequence/number system. The standard sequence/number system given in Method 3 was actually the system used in Europe before it got replaced by the Hindu number system. It is called the **Roman number system.** It was widely used in Europe for centuries, and was convenient for many purposes, but had the similar drawback that one cannot write arbitrarily large numbers without introducing more and more symbols. We will learn more about this system of writing numbers later on.

As illustrated by the three methods, history gives us examples of number systems formed using physical objects (such as sticks, pebbles, body parts, etc.), names, and written symbols. Some groups of people had numbers represented both by physical objects as well as by names, while others like the Chinese had all three forms of representation. The symbols occurring in a written number system are called **numerals.** For example, 0, 1, 5, 36, 193, etc., are some of the numerals occurring in the Hindu number system. Numerals representing 'smaller' numbers always had names, and so a number system composed of written symbols always went hand in hand with a number system composed of names, as is the case with the modern-day Hindu system.

Figure it Out

1. Suppose you are using the number system that uses sticks to represent numbers, as in Method 1. Without using either the number names or the numerals of the Hindu number system, give a method for adding, subtracting, multiplying and dividing two numbers or two collections of sticks. *Math Talk*

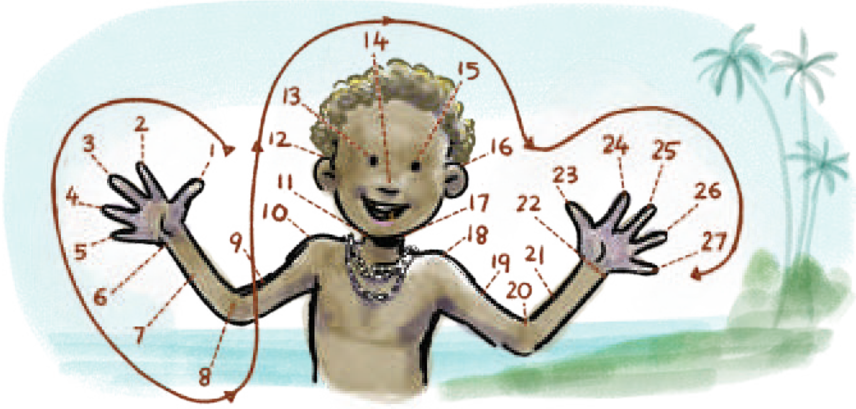
2. One way of extending the number system in Method 2 is by using strings with more than one letter—for example, we could use '*aa*' for 27. How can you extend this system to represent all the numbers? There are many ways of doing it! *Math Talk*

3. Try making your own number system. *Try This*

3.2 Some Early Number Systems

I. Use of Body Parts

Many groups of people across the world have used their hands and body parts for counting. Here is how a group of people in Papua New Guinea used and still use their body parts as the standard sequence/number system.



II. Tally Marks on Bones and Other Surfaces

One of the oldest methods of number representation is by making notches—marks cut on a surface such as a bone or a wall of a cave. These marks are also called **tally marks.**

In this method, a mark is made for each object that is being counted. So the final collection of marks represents the total number of objects. This method is very similar to the method of using sticks to count (Method 1), except for the fact that a mark is made instead of adding a stick.

Archaeologists have discovered bones dating back more than 20,000 years that seem to have tally marks. The oldest known such bones with markings that are thought to represent numbers are the Ishango bone and the Lebombo bone. The Ishango bone, dating back 20,000 to 35,000 years, was discovered in the Democratic Republic of Congo. It features notches arranged in columns, possibly indicating calendrical systems. The Lebombo bone, discovered in South Africa, is an even older tally stick with 29 notches, estimated to be around 44,000 years old. It is considered one of the oldest mathematical artefacts, and may have served as a tally stick or lunar calendar.

Lebombo bone

Ishango bone

III. Number Names Obtained by Counting in Twos

A group of indigenous people in Australia called the Gumulgal had the following words for their numbers.

Gumulgal
(Australia)
1. urapon
2. ukasar
3. ukasar-urapon
4. ukasar-ukasar
5. ukasar-ukasar-urapon
6. ukasar-ukasar-ukasar

? Can you see how their number names are formed?

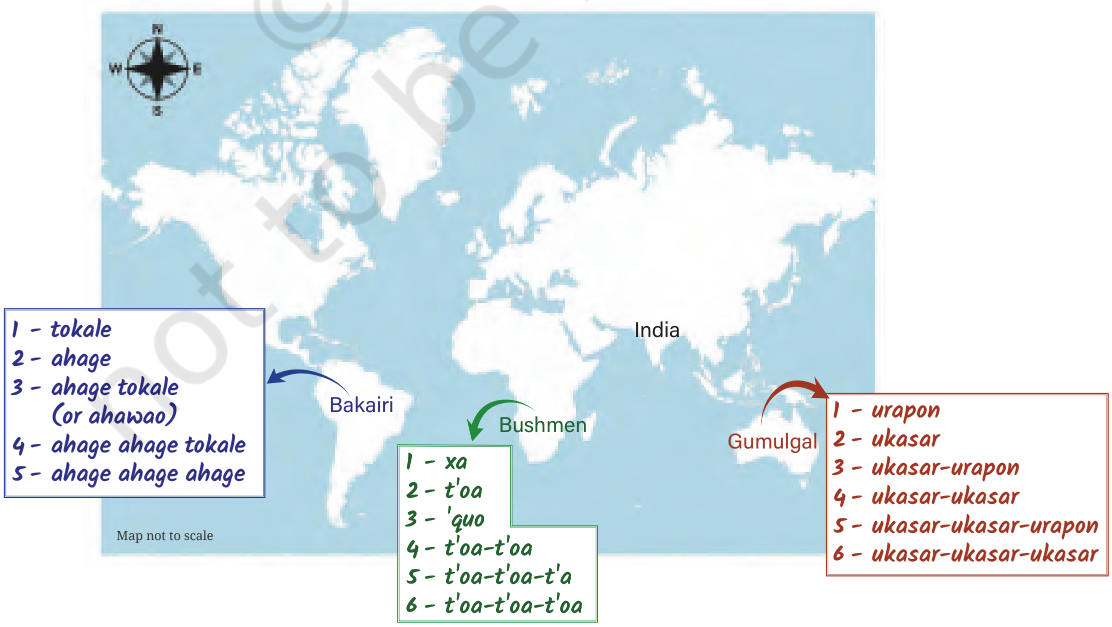
The number name for 3 is composed of number names of 2 and 1.

The number name for 4 is composed of two occurrences of the number name for 2.

? Can you see how the names of the other numbers are formed?

The numbers are counted in 2s, using which the number names are formed: 3 = 2 + 1, 4 = 2 + 2, 5 = 2 + 2 + 1, 6 = 2 + 2 + 2.
Gumulgal called any number greater than 6 *ras*.

There is a very interesting and puzzling historical phenomenon associated with this number system. Look at the following number systems of a group of indigenous people in South America, and the Bushmen of South Africa:



Despite being so far apart geographically, and with no trace of contact between them, these three groups have developed equivalent number systems! Historians have wondered how this happened. One theory is that these three groups of people may have had common ancestors, who used this number system. In course of time, their descendants migrated to these places.

Even though the number system of Gumulgal had number names for numbers only till 6, we can see the emergence of an idea here. Counting in 2s is more efficient for representing numbers than, for example, a tally system. A general form which this idea has taken in different number systems is as follows: count in groups of a certain number (like 2 in the case of Gumulgal's system), and use the word or symbol associated with this group size to represent bigger numbers. Some of the commonly used group sizes in different number systems have been 2, 5, 10 and 20. You can find the idea of counting by 5s in the Roman system (Table 1).

> This idea of counting in a certain group size and using it to represent numbers is an important idea in the history of the evolution of number systems.

One of the phenomena that could have led people to this idea might be the human limit for immediately knowing the size of a collection at a glance. Let us try out the following activity.

? Quickly count the number of objects in each of the following boxes:

Up to what group size could you immediately see the number of objects without counting? Most humans find it difficult to count groups having 5 or more objects in a single glance.

This limit of perception could have prompted people using tally marks to replace every group of, say, 5 marks, with a new symbol, as seen in the system shown in Table 1.

Math Talk

? What could be the difficulties with using a number system that counts only in groups of a single particular size? How would you represent a number like 1345 in a system that counts only by 5s?

Even though counting in groups of a particular size and using it for number representation is more efficient than the tally system, this method can still become cumbersome for larger numbers. The next system shows a refinement of this idea.

IV. The Roman Numerals

We have already seen the Roman number system till 20 (Table 1). We have seen that it uses I for 1, V for 5, X for 10.

To get the Roman numeral for any number till 39, it is first grouped into as many 10s as possible, the remaining is grouped into as many 5s as possible, and finally the remaining is grouped into 1s.

Example: Let us take the number 27.

$$27 = 10 + 10 + 5 + 1 + 1$$

So, 27 in Roman numerals is XXVII.

Instead of representing 50 as XXXXX, a new symbol is given to it: L. Following the way the number 4 is represented as 1 less than 5 — that is, as IV — 40 is represented as 10 less than 50 — that is, as XL. However, people using this system were not always consistent with this practice. Sometimes, 40 was also represented as XXXX.

The Roman number system introduces newer symbols to represent certain bigger numbers. Let us call all these numbers that have a new basic symbol as **landmark numbers**. Here are some of the landmark numbers of the Roman system and their associated numerals.

I	V	X	L	C	D	M
1	5	10	50	100	500	1,000

These symbols are used to denote other numbers as well. For example, consider the number 2367. Writing it as a sum of landmark numbers starting from 1000 such that we take as many 1000s as possible, 500s as possible, and so on, we get

$$2367 = 1000 + 1000 + 100 + 100 + 100 + 50 + 10 + 5 + 1 + 1$$

So in Roman numerals, this number is MMCCCLXII.

Figure it Out

1. Represent the following numbers in the Roman system.

   (i) 1222 (ii) 2999 (iii) 302 (iv) 715

We see how vastly efficient this system is compared to some of the previous number systems that we have seen. This system seems to have evolved out of the ancient Greek number system in around the 8th century BCE in Rome, and evolved over time. It spread throughout Europe with the expansion of the Roman empire.

> The efficiency of this system is due to the grouping of a given number by not just one group size, but a sequence of group sizes that we call landmark numbers, and then using these landmark numbers to represent the given number. This idea is the next important breakthrough in the history of the evolution of number systems.

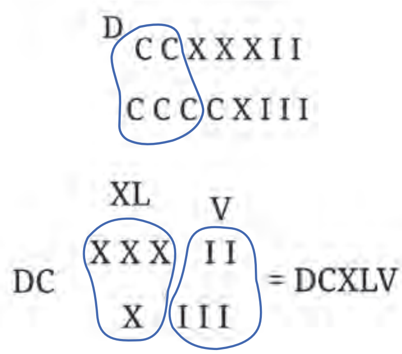
Despite the relative efficiency of the Roman system, it doesn't lend itself to an easy performance of arithmetic operations, particularly multiplication and division.

Example: Try adding the following numbers without converting them to Hindu numerals:

(a) CCXXXII + CCCCXIII

Let us find the total number of Is, Xs, and Cs, and group them starting from the largest landmark number.

Apparently, it looks like the largest landmark number is C, but note that 5 Cs (100s) make a D (500). So the sum is



Do it yourself now:

(b) LXXXVII + LXXVIII

How will you multiply two numbers given in Roman numerals, without converting them to Hindu numerals? Try to find the product of the following pairs of landmark numbers: V × L, L × D, V × D, VII × IX.

Try This

People using the Roman system made use of a calculating tool called the ***abacus*** to perform their arithmetic operations. We will see what it is in a later section. However, only specially trained people used this tool for calculation.

While going through the number systems discussed above, it should not be understood that, historically, one system developed as an improvement over the previous system. This point should be kept in mind when studying the upcoming number systems too. The actual story of how each of the number systems developed is much more complex, and many times not clearly known, and so we will not try to trace this in the chapter.

Figure it Out

1. A group of indigenous people in a Pacific island use different sequences of number names to count different objects. Why do you think they do this? *Math Talk*

2. Consider the extension of the Gumulgal number system beyond 6 in the same way of counting by 2s. Come up with ways of performing the different arithmetic operations (+, –, ×, ÷) for numbers occurring in this system, without using Hindu numerals. Use this to evaluate the following: *Math Talk*

(i) (ukasar-ukasar-ukasar-ukasar-urapon) + (ukasar-ukasar-ukasar-urapon)

(ii) (ukasar-ukasar-ukasar-ukasar-urapon) – (ukasar-ukasar-ukasar)

(iii) (ukasar-ukasar-ukasar-ukasar-urapon) × (ukasar-ukasar)

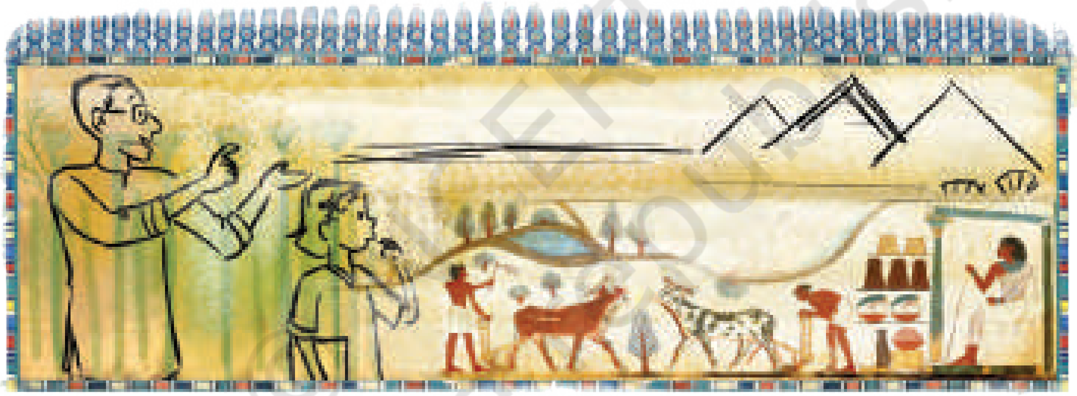
(iv) (ukasar-ukasar-ukasar-ukasar-ukasar-ukasar-ukasar-ukasar) ÷ (ukasar-ukasar)

3. Identify the features of the Hindu number system that make it efficient when compared to the Roman number system. *Math Talk*

4. Using the ideas discussed in this section, try refining the number system you might have made earlier. *Try This*

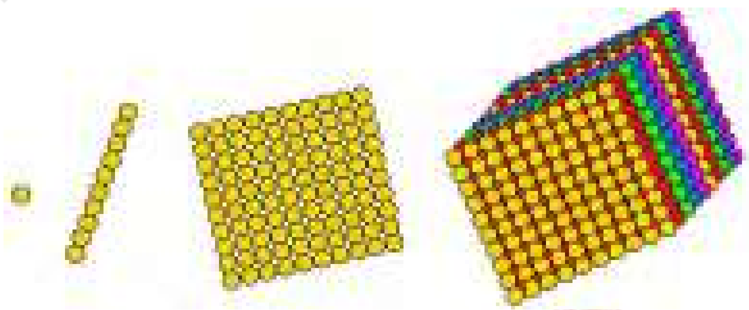
3.3 The Idea of a Base

I. The Egyptian Number System

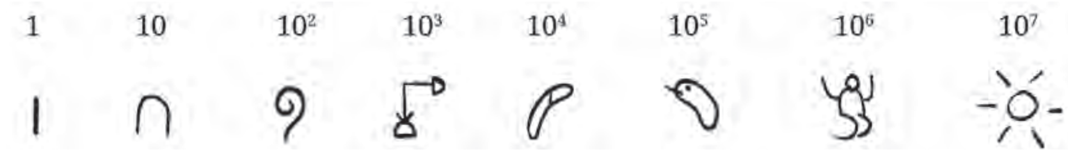


We are now going to see a written number system that the Egyptians developed around 3000 BCE. In this system, we see the use of landmark numbers to group and represent a given number. However, what makes this system special is its sequence of landmark numbers.

Imagine making collections of pebbles. The first landmark number is 1. Group together 10 collections of the previous landmark number (1). Its size is the second landmark number which is 10. Group together 10 collections of the previous landmark number (10). Its size is the third landmark number which is 10 × 10 = 100, and so on.



Each landmark number is 10 times the previous one. Since 1 is the first landmark number, they are all powers of 10. The following are the symbols given to these numbers —



As in the case of Roman numbers, a given number is counted in groups of the landmark numbers, starting from the largest landmark number less than the given number. This is then used to assign the numeral.

For example 324 which equals 100 + 100 + 100 + 10 + 10 + 4 is written as .

Figure it Out

1. Represent the following numbers in the Egyptian system: 10458, 1023, 2660, 784, 1111, 70707.

2. What numbers do these numerals stand for?

(i) 

(ii) 

II. Variations on the Egyptian System and the Notion of Base

Instead of grouping together 10 collections of size equal to the previous landmark number (as in the case of the Egyptian system), can we get a number system by grouping together 5 collections of size equal to the previous landmark number? Can this 5 be replaced by any positive integer?

Let us examine this possibility. Let 1 be the first landmark number.

Group together 5 collections of size equal to the previous landmark number (1). Its size is the second landmark number which is 5.

Group together 5 collections of size equal to the previous landmark number (5). Its size is the third landmark number which is 5 × 5 = 25.

Group together 5 collections of size equal to the previous landmark number (5). Its size is the fourth landmark number which is 5 × 25 = 125.

Thus, we have a new number system where each landmark number is 5 times the previous one. Since 1 is the first landmark number, they are all powers of 5.

$5^0 = 1$ $5^1 = 5$ $5^2 = 25$ $5^3 = 125$ $5^4 = 625$ $5^5 = 3125$

Express the number 143 in this new system.

Let us start grouping, starting with the size $5^3 = 125$, as this is the largest landmark number smaller than 143. We get—

$$143 = 125 + 5 + 5 + 5 + 1 + 1 + 1.$$

So the number 143 in the new system is ○□□□△△△.

Number systems having landmark numbers in which the

(a) first landmark number is 1, and

(b) every next landmark number is obtained by multiplying the current landmark number by some fixed number n is said to be a **base-*n*** number system.

The Egyptian number system is a base-10 system, and the number system that we created is a base-5 system. A base-10 number system is also called a **decimal number system.**

Figure it Out

1. Write the following numbers in the above base-5 system using the symbols in Table 2: 15, 50, 137, 293, 651.

2. Is there a number that cannot be represented in our base-5 system above? Why or why not?

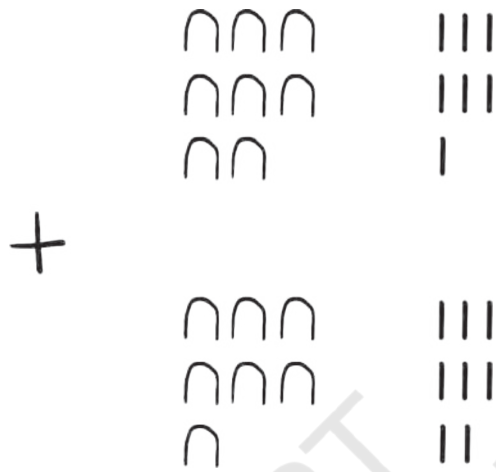
3. Compute the landmark numbers of a base-7 system. In general, what are the landmark numbers of a base-n system?

The landmark numbers of a base-n number system are the powers of n starting from $n^0 = 1, n, n^2, n^3, ...$

Advantages of a Base-*n* System

What is the advantage of having landmark numbers that are all the powers of a number? To understand this, let us perform some arithmetic operations using them.

Example: Add the following Egyptian numerals:

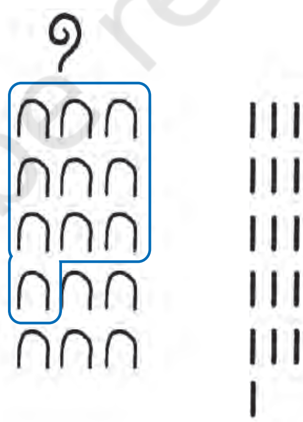


Let us find the total number of | and ∩ and group them starting from the largest possible landmark number. It has a total of—

15 ∩ and 15 |.

Since 10 ∩ gives the next landmark number 𓍢, the sum can be regrouped as—

Sum =



Since 10 | gives a ∩, we have

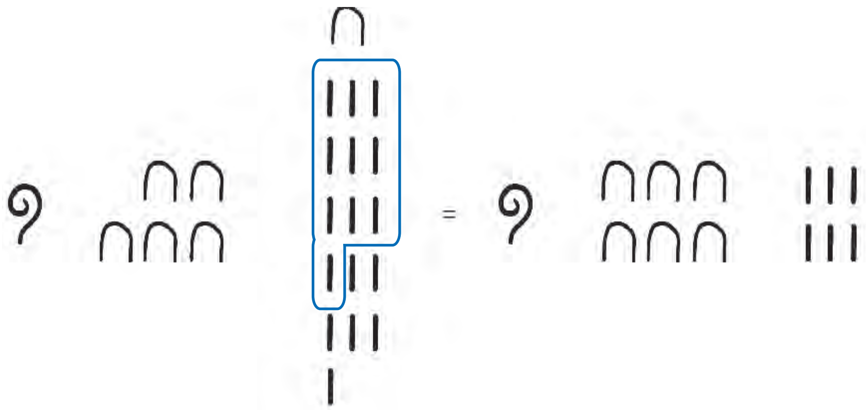
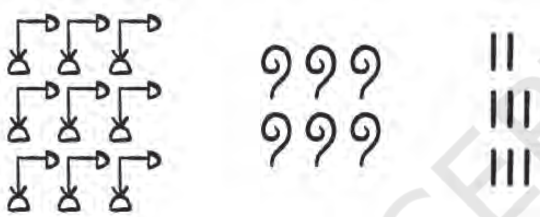
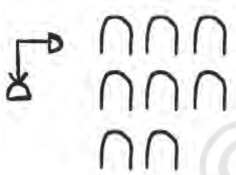
Sum = 

Figure it Out

1. Add the following Egyptian numerals:

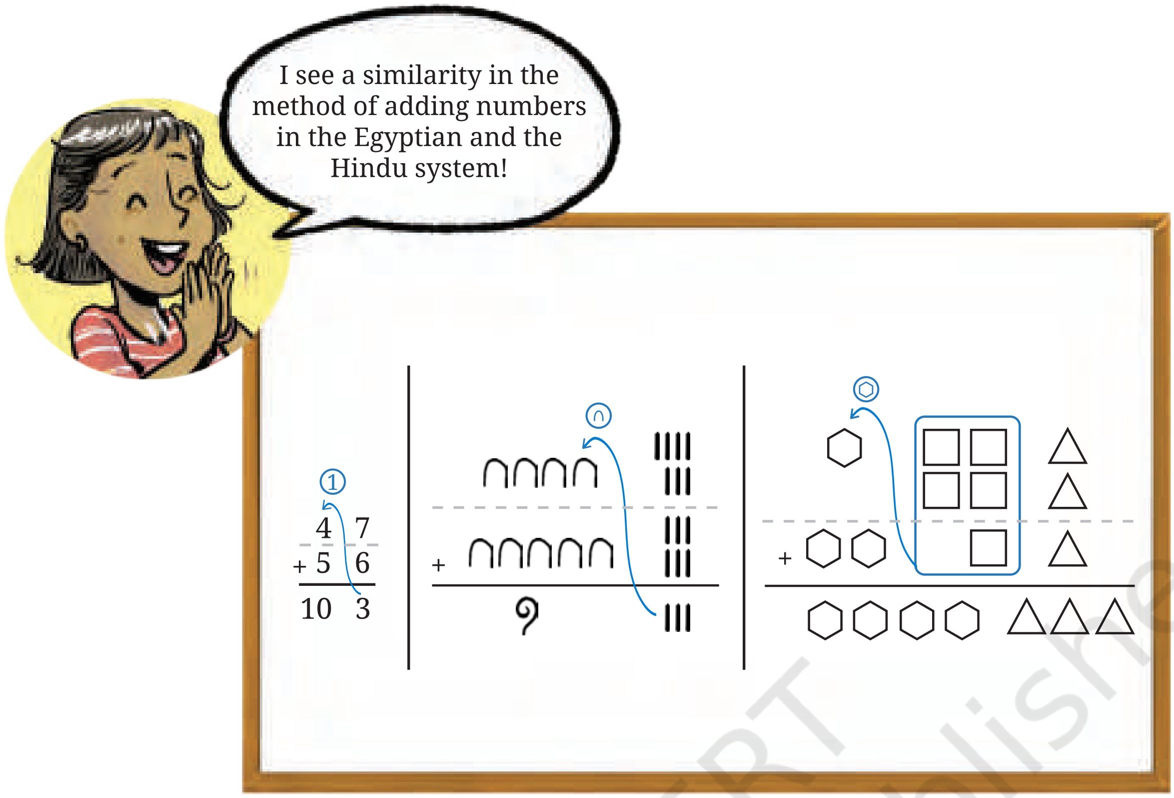
(i)  and 

(ii)  and 

2. Add the following numerals that are in the base-5 system that we created:

○ ⬡ ⬡ □ △ △ + ○ ○ ○ ⬡ □ □ △ △

Remember that in this system, 5 times a landmark number gives the next one!



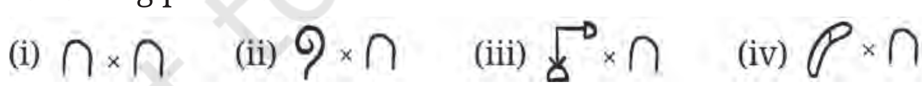
Contrast the addition done in a base-n number system with that done in the Roman system. In the Roman system, the grouping and rearranging has to be done carefully as it is not always by the same size that each landmark number has to be grouped to get the next one.

The advantage of a number system with a base becomes more evident when we consider multiplication.

? How to multiply two numbers in Egyptian numerals?

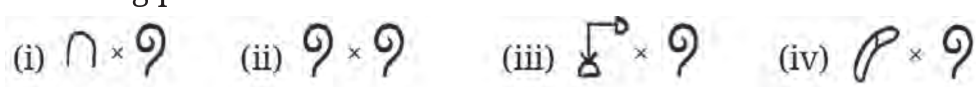
Let us first consider the product of two landmark numbers.

? 1. What is any landmark number multiplied by ∩ (that is 10)? Find the following products—

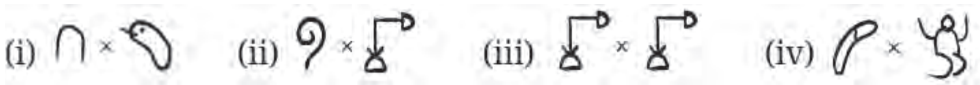


Each landmark number is a power of 10 and so multiplying it with 10 increases the power by 1, which is the next landmark number.

? 2. What is any landmark number multiplied by 𓍢 (10^2)? Find the following products—



? Find the following products—



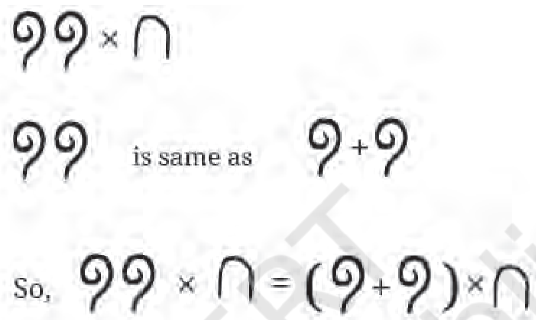
Thus, the product of any two landmark numbers is another landmark number!

? Does this property hold true in the base-5 system that we created? Does this hold for any number system with a base?

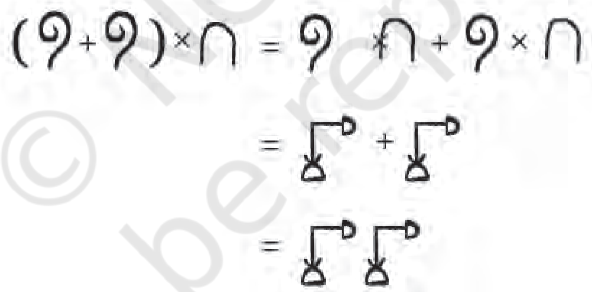
Math Talk

? What can we conclude about the product of a number and ∩ (10), in the Egyptian system?

(i)



As these are numbers, the distributive law holds. So,



(ii) 𓍢∩∩| × ∩

𓍢∩∩| is the same as 𓍢 + ∩∩ + |. Thus,

𓍢∩∩| × ∩ = (𓍢 + ∩∩ + |) × ∩

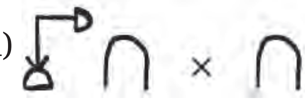
Will the distributive property hold here? For the same reason that it holds for $(a + b) \times n$, it also holds when one of the numbers has more than 2 terms. For example, $(a + b + c) \times n = an + bn + cn$. So,

(𓍢 + ∩∩ + |) × ∩ = (𓍢 × ∩) + (∩∩ × ∩) + (| × ∩)

= 𓆼 + 𓍢𓍢 + ∩

= 𓆼 𓍢𓍢 ∩

Now find the following products—

(i)  (ii) 

What would be a simple rule to multiply a number with ∩?

As has been seen, a process of multiplying two numbers involves the multiplication of landmark numbers. When the landmark numbers are powers of a number, then their product is another landmark number. This fact simplifies the process of multiplication. However, this is not the case with the Roman numerals, which is why multiplication using them is difficult.

Thus, a number system whose landmark numbers are powers of a number, i.e., a number system with a base, is efficient not only in number representation but also in its utility in carrying out arithmetic operations.

The idea of a number system with a base was a turning point in the history of the evolution of number systems. Our modern Hindu number system is built on this structure.

Abacus that Makes Use of the Decimal System

In around the 11th century, even the people still using the Roman numerals started using a calculating device—the abacus—constructed using a decimal system. It was a board with lines, as shown in the Fig. 3.1. Starting from the line that stood for 1, each successive line stood for a successive power of 10.

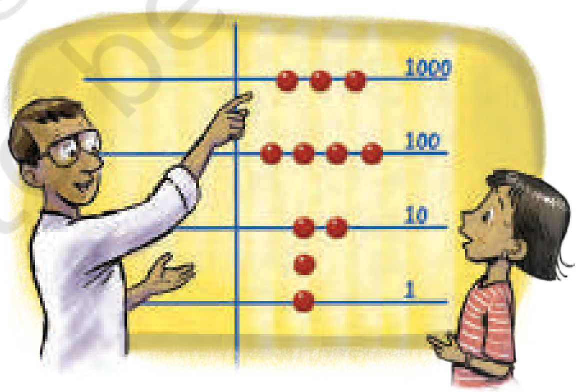


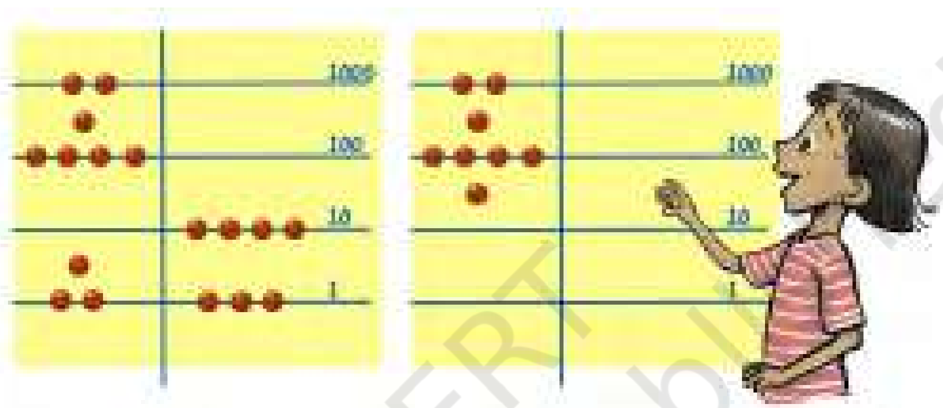
Fig. 3.1: Abacus

Numbers were represented in it as follows: the given number was first grouped into the landmark numbers (powers of 10), in exactly the same way we have been grouping them so far. For each power of 10, as many counters were placed on its line as the number of times it occurred in the grouping. The presence of a counter above a line contributed a value of 5.

For example let us take the number 3426. It can be grouped as
3426 = 1000 + 1000 + 1000 + 100 + 100 + 100 + 100 + 10 + 10+ 1 + 1 + 1 + 1 + 1 + 1

This number was represented as shown in Fig. 3.1. Notice how the 6 ones are represented.

To get an idea of how the abacus was used for calculations, let us consider a simple addition problem: 2907 + 43. The two numbers were taken on either side of the vertical partition.



How would you use this to find the sum?

The counters along each line were brought together. What is to be done if the total in a line exceeded 10?

Hint: In this problem, the 7 ones and the 3 ones together make 10 ones which contributes a counter to the line representing 10s.

III. Shortcomings of the Egyptian System

Despite being a number system that enabled relatively efficient number representations for numbers till a crore (10^7), and relatively easy computations, the Egyptian system had a drawback.

If larger and larger numbers needed to be represented, then there was a need for inventing an unending sequence of symbols for higher and higher powers of 10. Here we see the original challenge of number representation resurfacing in a different form!

The next and the final idea in the history of the evolution of number systems not only solves this problem but also remarkably simplifies number representation and computations!

Figure it Out

1. Can there be a number whose representation in Egyptian numerals has one of the symbols occurring 10 or more times? Why not? Math Talk

2. Create your own number system of base 4, and represent numbers from 1 to 16.

3. Give a simple rule to multiply a given number by 5 in the base-5 system that we created.

3.4. Place Value Representation

I. The Mesopotamian Number System

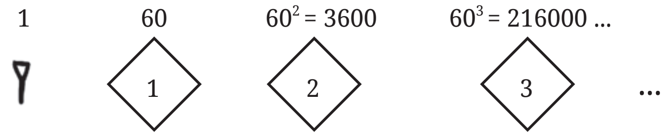
In the beginning, the number system used in ancient Mesopotamia had different symbols for different landmark numbers. In later times, it became a base-60 system, also called the **sexagesimal system**, with a very efficient number representation.

It has puzzled many why they chose base 60. Different theories exist to explain this, ranging from the connection between 60 and the periods of some important events (like the length of their lunar month which had 30 days, or the time taken for the Sun to complete one revolution around the Earth when Earth is taken to be stationary), the ease of representing fractions (we will not go into this here), their earlier sequence of landmark numbers — 1, 10, 60, 600, 3600, 36000,... — getting reduced to only the powers of 60, and so on.

The influence of the Mesopotamian sexagesimal system, also known as the **Babylonian number system**, can be seen even now in our units of time measurements — 1 hour = 60 minutes and 1 minute = 60 seconds. This system used the symbol 𒁹 for 1 and 𒌋 for 10.

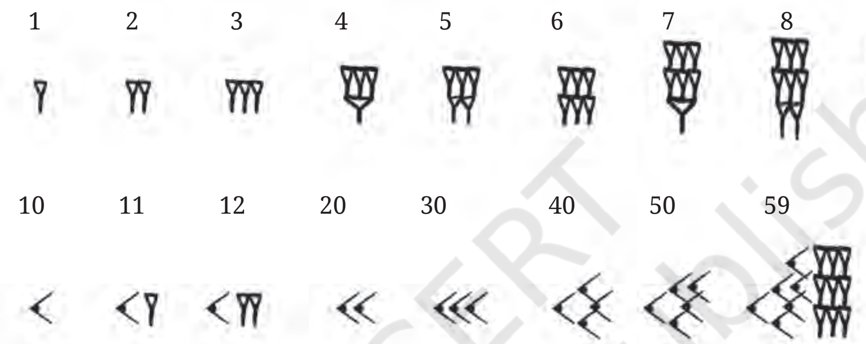
Let us now briefly pause on the study of their number system, and ideate on how one can build an efficient number system using the Mesopotamian features seen so far.

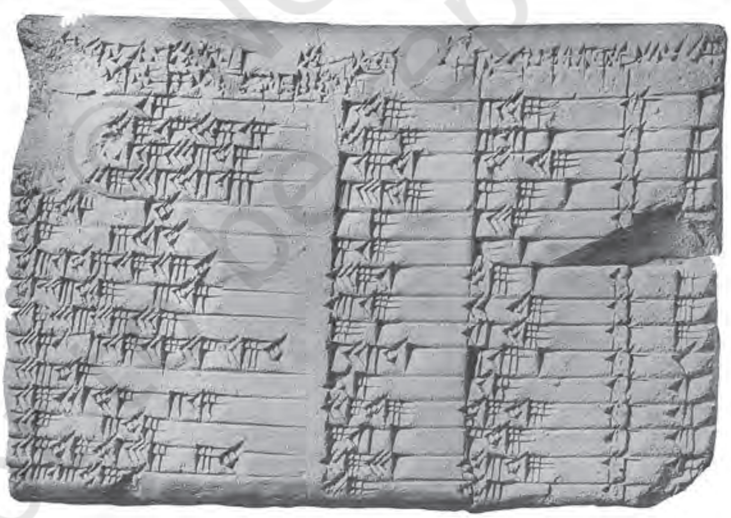
Let us give our own symbols to their landmark numbers—



Note that we have actually used Indian numerals in creating these symbols. We could have invented our own symbols but for the sake of easy recall and use, we have chosen to take help of the familiar numerals 1, 2, 3, ...

Using  and , numbers from 1 to 59 can be represented—



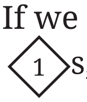


Reproduction of a Mesopotamian Tablet

Example: Let us represent the number 640 in this system.

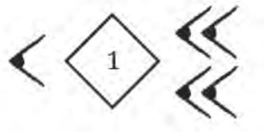
Grouping it into landmark numbers, we see that

$$640 = (10) \times 60 + 40.$$

If we use the Egyptian idea, this number would be represented using 10 s, and 40 would be represented using 4 s.

? Can we represent this more compactly?

We can simply represent this number as



which can be read as ten 60s and one 40, just as we have written in the equation.

? **Example:** Let us try another number — 7530.

$$7530 = (2) \times 3600 + (5) \times 60 + 30$$

So, its representation would be

Note that when a number is grouped into powers of 60 for its representation, no power of 60 can occur 60 or more times. If this happens, then 60 of them can be grouped to form the next power of 60. For example, consider the expression—

$$\begin{aligned}(1) \times 3600 + (70) \times 60 + 2 &= (1) \times 60^2 + (60 + 10) \times 60 + 2\\ &= (1) \times 60^2 + 60^2 + (10) \times 60 + 2\\ &= (2) \times 60^2 + (10) \times 60 + 2\end{aligned}$$

Therefore, any number can be represented using the numerals from 1–59, along with the numerals for landmark numbers.

Now, what if we make the representation even more compact by dropping the symbols for the different powers of 60 altogether?

	640	7530
Our earlier representation		
Our compact representation		

This is exactly what the Mesopotamians did! In their numeral, the rightmost set of symbols showed the number of 1s, the set of symbols to its left showed the number of 60s, the next showed the number of 3600s and so on. Whenever there was no occurrence of a power of 60, a blank space was given in that position.

It does not seem that the Mesopotamians arrived at this idea in the same way we did. Some scholars suggest that the similarity of symbols given to the landmark numbers 1 and 60 in their earlier number system, and an accidental usage of them, might have made them stumble upon this idea.

Figure it Out

1. Represent the following numbers in the Mesopotamian system —

   (i) 63 (ii) 132 (iii) 200 (iv) 60 (v) 3605

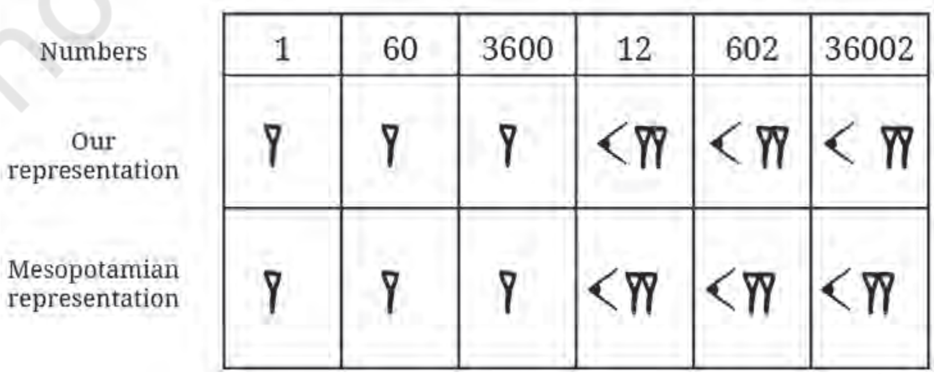
Thus, we can see how the Mesopotamian system removes the need for generating an unending sequence of symbols for the landmark numbers by making use of the positions where the symbols are written. Such a number system (having a base) that makes use of the position of each symbol in determining the landmark number that it is associated with is called a **positional number system** or a **place value system.**

This idea of place value marks the highest point in the history of evolution of number systems, and gives a very elegant solution to the problem of representing the unending sequence of numbers using only a finite number of different symbols!

The Mesopotamian system however cannot be considered a fully developed place value system. It has certain defects that lead to confusion while reading a number.

? Look at the representation of 60. What will be the representation for 3,600?

While writing the numerals, the spacing between symbols was not given the way we are giving it here. It was also difficult to maintain a consistent spacing for blanks across different manuscripts written by different people. These created ambiguities. For example, consider the representations of the following numbers —



Because of the ambiguity in finding which symbols correspond to which powers of 60, the same numeral can be read in different ways. Even in our representation which uses uniform spacing between symbols for different powers of 60, it is difficult to know the number of blanks between two sets of symbols, as in the representation of 36002.

To address the issue arising out of blank spaces, the later Mesopotamians used a brilliant idea of assigning a 'placeholder' symbol < to denote a blank space. This is like the 0 (zero) we use in our system. Thus, zero—the symbol that shows nothingness—is indispensable as a placeholder in a place value system in which numbers are written in an unambiguous manner.

Even with the problem arising out of blank spaces solved, other ambiguities still remained in the system. For example, the placeholder symbol was primarily used in the middle of numbers and not at the end; so they would not use it to represent a number like (what we would write as) 3600.

II. The Mayan Number System

In Central America, there flourished a civilisation known as the Mayan civilisation that made great intellectual and cultural progress between the 3rd and 10th centuries CE. Among their intellectual achievements stands their place value system designed independently of those in Asia. They also made use of a placeholder symbol, for the modern-day '0', that looked like a seashell.

MAYAN NUMBER SYSTEM

Almost a base-20 system

LANDMARK NUMBERS

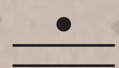
1, 20, 20×18 = 360, 20^2×18 = 7200, 20^3×18 = 144000

SYMBOLS  is 0 • is 1 —— is 5

Symbols in the Mayan Number system are placed vertically to represent a number.

A numeral 

How is this to be read ?

360	• • • •	4
20		11
1		0
Landmark number positions	Vertically placed symbols	Meaning of the symbols

= (4)×360 + (11)×20 + (3)×10

= 1660

Here we find a puzzling phenomenon. Why was their third landmark number 360 rather than 400? Some scholars feel that this might have something to do with their calendars.

They used a dot for 1, and a bar for 5. These were used to denote numbers from 1 to 19.

The symbols associated with different landmark numbers were written one below the other with the lowermost set of symbols corresponding to the number of 1s, the set above corresponding to the number of 20s, the set above to the number of 360s and so on.

? Represent the following numbers using the Mayan system:

(i) 77 (ii) 100 (iii) 361 (iv) 721

Because the Mayan system is not an actual base-20 system, it lacks the advantages that a system with a base has for computations. Nevertheless, their place value notation and their use of a placeholder symbol for zero is considered an important advance in the history of number systems.

A curious fact is that we can still find the use of base-20 in the number names of some European languages.

III. The Chinese Number System

The Chinese used two number systems — a written system for recording quantities, and a system making use of rods for performing computations. The numerals in the rod-based number system are called **rod numerals.** Here we discuss the rod numerals, which are more efficient in writing and computing with numbers than the written system of the Chinese.

The rod numerals developed in China by at least by 3rd century AD and were used till the 17th century. It was a decimal system (base-10). The symbols for 1 to 9 were as follows:

CHINESE NUMBER SYSTEM

Base-10 or Decimal

	1	2	3	4	5	6	7	8	9
Zongs	𝍠	𝍡	𝍢	𝍣	𝍤	𝍥	𝍦	𝍧	𝍨
Hengs	𝍩	𝍪	𝍫	𝍬	𝍭	𝍮	𝍯	𝍰	𝍱

Note: The *zongs* represent units, hundreds, tens of thousands, etc., and the *hengs* tens, thousands, hundreds of thousands, etc.

A numeral 𝍪 𝍥 𝍫 𝍣

How is this to be read?	2 (*Heng*) 𝍪	6 (*Zong*) 𝍥	3 (*Heng*) 𝍫	4 (*Zong*) 𝍣
Landmark number positions	10^3	10^2	10	1

$= (2) \times 10^3 + (6) \times 10^2 + (3) \times 10 + (4) \times 1$

$= 2634$

数学

Like the Mesopotamians, the rod numerals used a blank space to indicate the skipping of a place value. However, because of the slightly more uniform sizes of the symbols for one through nine, the blank spaces were easier to locate than in the Mesopotamian system.

Notice how similar the rod numerals are to the Hindu system. The Chinese system, with a symbol for zero, would be a fully developed place value system.

IV. The Hindu Number System

? Where does the Hindu/Indian number system figure in the evolution of ideas of number representation? What are its landmark numbers? And does it use a place value system?

Hindu Number System

Base-10 or Decimal

Ten symbols 0, 1, 2, 3, 4, 5, 6, 7, 8, 9

A numeral 375

How is this to be read?	3	7	5
Landmark number positions	10^2	10	1

$$= (3) \times 10^2 + (7) \times 10 + (5) \times 1$$

$$= 375$$

As can be seen, the Hindu number system is a place value system. The Hindu number system has had a symbol for 0 at least as early as 200 BCE. Because of the use of 0 as a digit, and the use of a single digit in each position, this system does not lead to any kind of ambiguity when reading or writing numerals. It is for this reason that the Hindu number system is now used throughout the world.

The use of 0 as a digit, and indeed as a number, was a breakthrough that truly changed the world of mathematics and science. In Indian mathematics, indeed, zero was not just used as a placeholder in the place value system, but was also given the status of a number in its own right, on par with other numbers. The arithmetic properties of the number 0 (e.g., that 0 plus any number is the same number, and that 0 times any number is zero) were explicitly used by Aryabhata in his *Āryabhaṭīya* in 499 CE to compute with and do elaborate scientific computations using Hindu numerals. The use of 0 as a number like any other number, on which one can perform the basic arithmetic operations, was codified by Brahmagupta in his work *Brāhmasphuṭasiddhānta* in 628 CE, as we learned in an earlier grade.

By introducing 0 as a number, along with the negative numbers, Brahmagupta created what in modern terms is called a **ring**, i.e., a set of numbers that is closed under addition, subtraction, and multiplication (i.e., any two numbers in the set can be added, subtracted, or multiplied to get another number in that set). These new ideas laid the foundations for modern mathematics, and particularly for the areas of algebra and analysis.

Hopefully, this gives you a sense of all the ideas that went into writing and computing with numbers in the way that we do today. The discovery of 0 and the resulting Indian number system is truly one of the greatest, most creative, and most influential inventions of all time—appearing constantly in our daily lives and forming the basis of much of modern science, technology, computing, accounting, surveying, and more. The next time you are writing numbers, think about the incredible history behind them and all the deep ideas that went into their discovery!

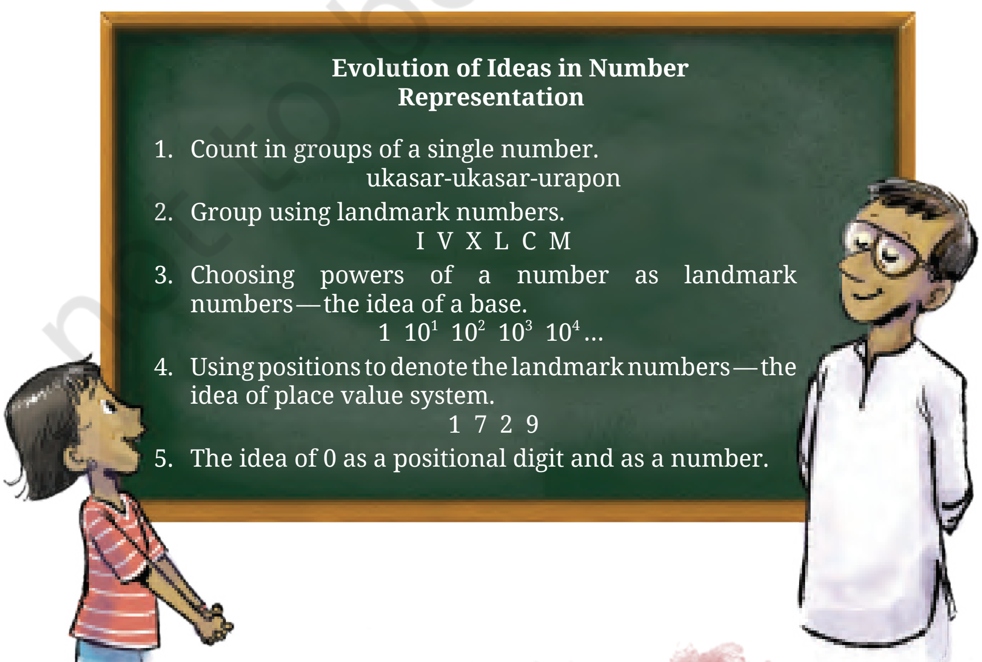


Figure it Out

1. Why do you think the Chinese alternated between the *Zong* and *Heng* symbols? If only the *Zong* symbols were to be used, how would 41 be represented? Could this numeral be interpreted in any other way if there is no significant space between two successive positions?

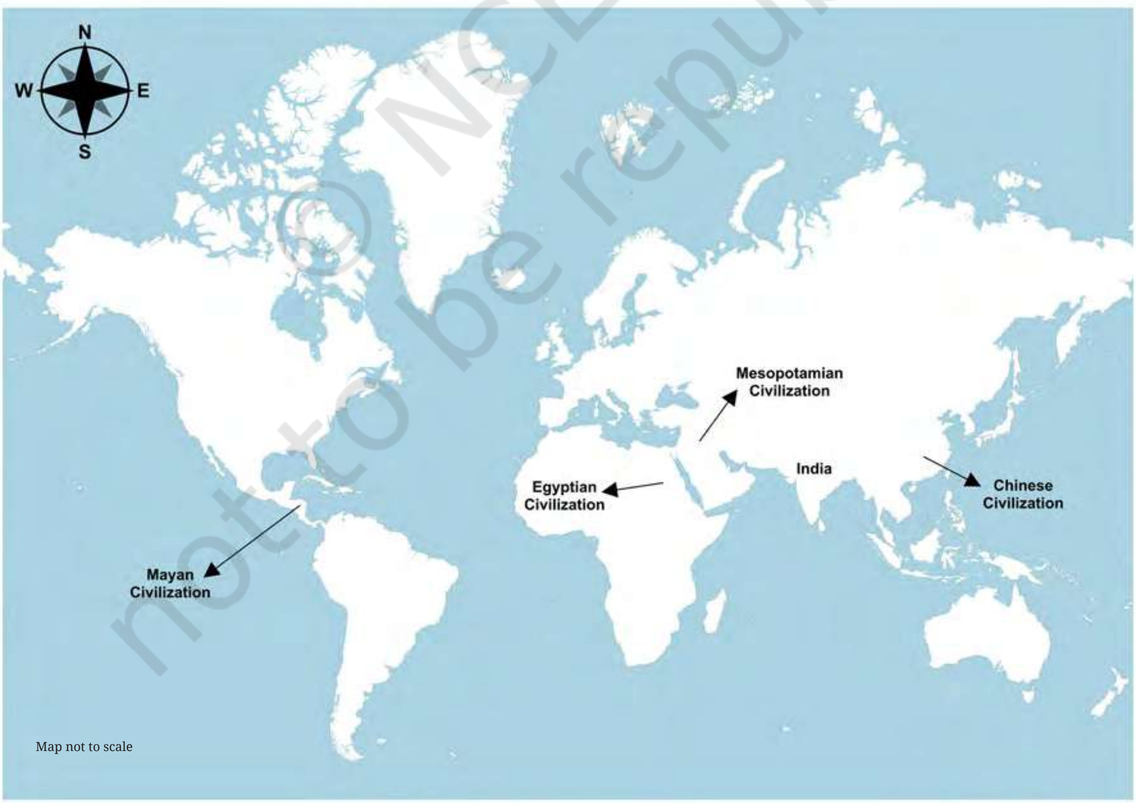
   Math Talk

2. Form a base-2 place value system using '*ukasar*' and '*urapon*' as the digits. Compare this system with that of the Gumulgal's.

3. Where in your daily lives, and in which professions, do the Hindu numerals, and 0, play an important role? How might our lives have been different if our number system and 0 hadn't been invented or conceived of?

4. The ancient Indians likely used base 10 for the Hindu number system because humans have 10 fingers, and so we can use our fingers to count. But what if we had only 8 fingers? How would we be writing numbers then? What would the Hindu numerals look like if we were using base 8 instead? Base 5? Try writing the base-10 Hindu numeral 25 as base-8 and base-5 Hindu numerals, respectively. Can you write it in base-2?

   Math Talk



The map shows the locations of the different civilisations. They existed in different time periods.

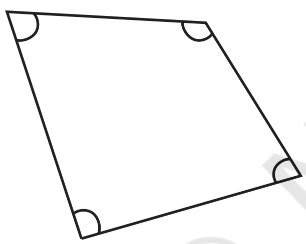
SUMMARY

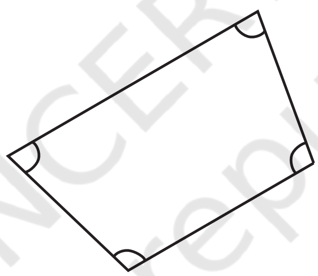
- To represent numbers, we need a standard sequence of objects, names, or written symbols that have a fixed order. This standard sequence is called a **number system.**
- The symbols representing numbers in a written number system are called **numerals.**
- In a number system, **landmark numbers** are numbers that are easily recognisable and used as reference points for understanding and working with other numbers. They serve as anchors within the number system, helping people to orient themselves and make sense of quantities, particularly larger ones.
- A number system whose landmark numbers are the powers of a number n is referred to as a **base-*n* number system.**
- Number systems having a base that make use of the position of a symbol in determining the landmark number that it is associated with are called **positional number systems** or **place value systems.**
- Place value representations were used in the Mesopotamian (Babylonian), Mayan, Chinese and Indian civilisations.
- The system of numerals that we use throughout the world today is the **Hindu number system** (also sometimes called the **Indian number system,** or the **Hindu-Arabic number system).** It is a place value system with (usually) 10 digits, including the digit 0 which is treated on par with other digits. Due to its use of 0 as a number, the system enables the writing of all numbers unambiguously using just finitely many symbols, and also enables efficient computation. The system originated in India around 2000 years ago, and then spread across the world, and is considered one of human history's greatest inventions.

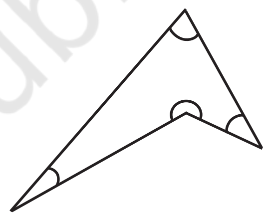
4 QUADRILATERALS

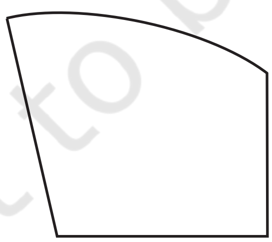
In this chapter, we will study some interesting types of four-sided figures and solve problems based on them. Such figures are commonly known as quadrilaterals. The word 'quadrilateral' is derived from Latin words — *quadri* meaning four, and *latus* referring to sides.

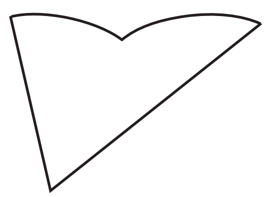
? Observe the following figures.


(i)


(ii)


(iii)


(iv)


(v)

Figs. (i), (ii), and (iii) are quadrilaterals, and the others are not. Why?

The angles of a quadrilateral are the angles between its sides, as marked in Figs. (i), (ii), and (iii).

We will start with the most familiar quadrilaterals—rectangles and squares.

4.1 Rectangles and Squares

We know what rectangles are. Let us define them.

Rectangle: A rectangle is a quadrilateral in which—

(i) The angles are all right angles (90°), and

(ii) The opposite sides are of equal length.

The definition precisely states the conditions a quadrilateral has to satisfy to be called a rectangle.

? Are there other ways to define a rectangle?

Let us consider the following problem related to the construction of rectangles.

A Carpenter's Problem

? A carpenter needs to put together two thin strips of wood, as shown in Fig. 1, so that when a thread is passed through their endpoints, it forms a rectangle.

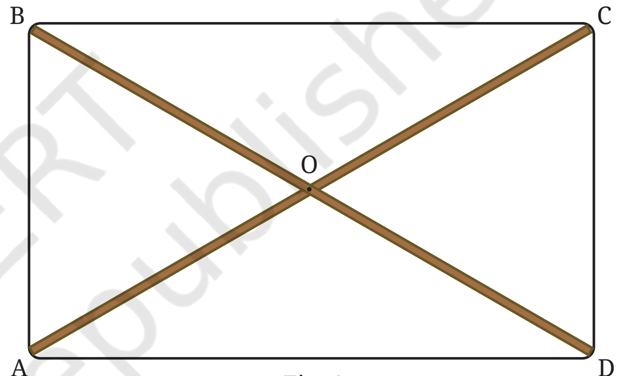


Fig. 1

She already has one 8 cm long strip. What should be the length of the other strip? Where should they both be joined?

Let us first model the structure that the carpenter has to make. The strips can be modelled as line segments. They are the diagonals of the quadrilateral formed by their endpoints. For the quadrilateral to be a rectangle, we need to answer the following questions —

? 1. What is the length of the other diagonal?

? 2. What is the point of intersection of the two diagonals?

? 3. What should the angle be between the diagonals?

? Let us answer these questions using geometric reasoning (deduction). If that is challenging, try to construct/measure some rectangles.

To find the answers to these questions, let us suppose that we have placed the diagonals such that their endpoints form the vertices of a rectangle, as shown in Fig. 2.

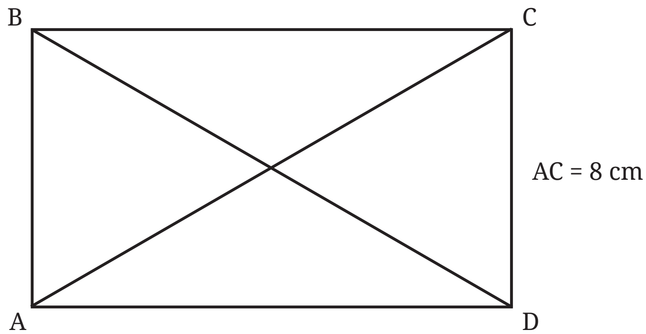
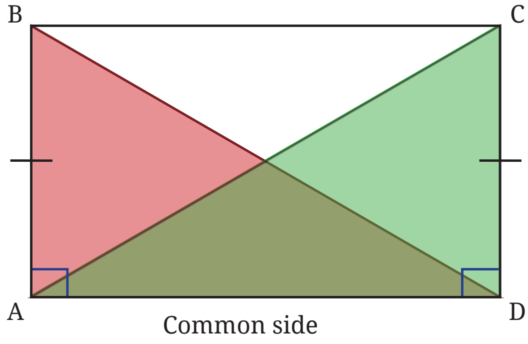


Fig. 2

Deduction 1— What is the length of the other diagonal?

This can be deduced using congruence as follows—



Since ABCD is a rectangle, we have

AB = CD
∠BAD = ∠CDA = 90°
AD is common to both triangles.

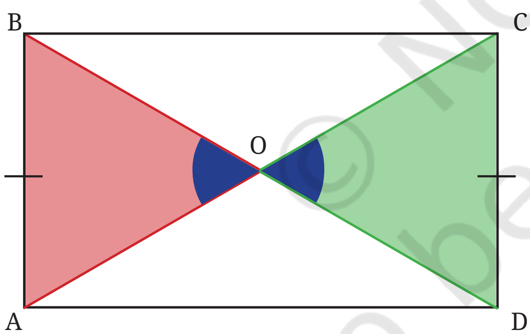
So, ΔADC ≅ ΔDAB by the SAS congruence condition.

Therefore, AC = BD, since they are corresponding parts of congruent triangles. This shows that the diagonals of a rectangle always have the same length.

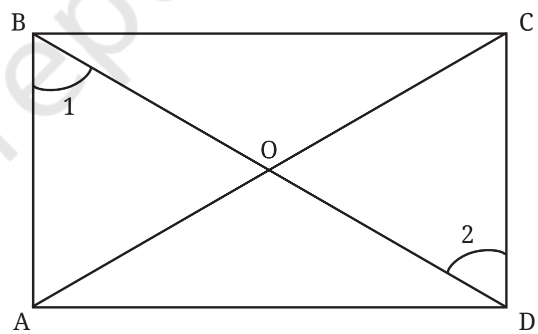
So the other diagonal must also be 8 cm long. You can verify this property by constructing/measuring some rectangles.

Deduction 2—What is the point of intersection of the two diagonals?

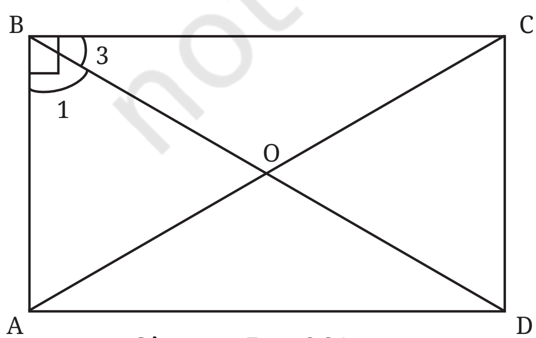
This can also be found using congruence. Since we need to know the relation between OA and OC, and OB and OD, which two triangles of the rectangle ABCD should we consider?



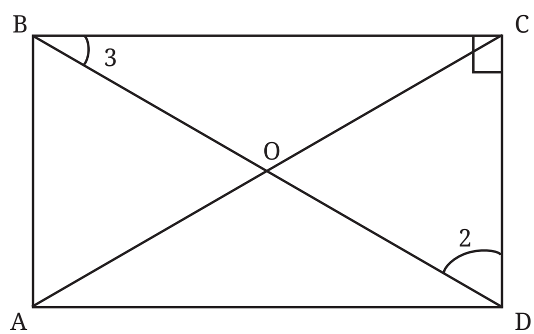
The blue angles are equal since they are vertically opposite angles.



In order to show congruence, consider ∠1 and ∠2. Are they equal?



Since ∠B = 90°,
∠3 + ∠1 = 90°.



In ΔBCD, since
∠3 + ∠2 + 90 = 180,
we have ∠3 + ∠2 = 90°.

So, ∠1 = ∠2 (= 90° – ∠3).
Thus, by the AAS condition for congruence, ΔAOB ≅ ΔCOD.

Hence OA = OC and OB = OD, since they are corresponding parts of congruent triangles. So, O is the midpoint of AC and BD.

This shows that **the diagonals of a rectangle always intersect at their midpoints.**

Therefore, to get a rectangle, the diagonals must be drawn so that they are equal and intersect at their midpoints.

When the diagonals cross at their midpoints, we say that the diagonals bisect each other. **Bisecting** a quantity means **dividing it into two equal parts.**

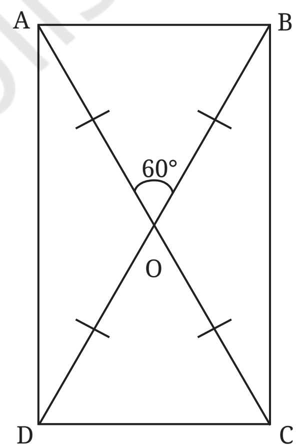
Verify this property by constructing some rectangles and measuring their diagonals and the points of intersection.

? Can the following equalities be used to establish that ΔAOD ≅ ΔCOB? *Math Talk*

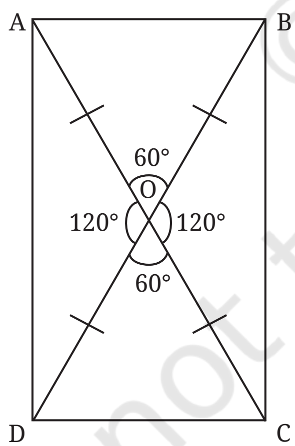
AO = CO (proved above)
∠AOB = ∠COD (vertically opposite angles)
AD = CB

Deduction 3— What are the angles between the diagonals?

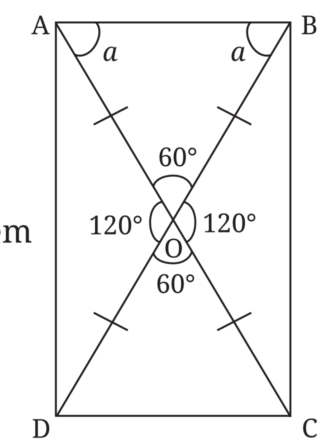
Let us check what quadrilateral we get if we draw the two diagonals such that their lengths are equal, they bisect each other and have an arbitrary angle, say 60°, between them as shown in the figure to the right.



? Can you find all the remaining angles?



We can find the remaining angles between the diagonals using our understanding of vertically opposite angles and linear pairs.

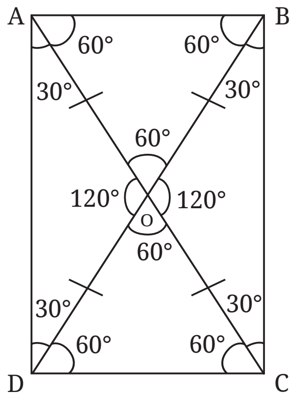


In ΔAOB, since OA = OB, the angles opposite them are equal, say *a*.

? Can you find the value of *a*?

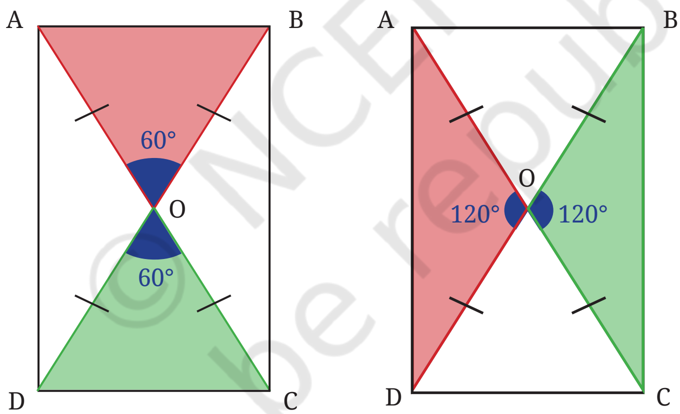
In ΔAOB, we have,
$a + a + 60 = 180$ (interior angles of a triangle).
Therefore $2a = 120$.
Thus $a = 60$.

Similarly, we can find the values of all the other angles.



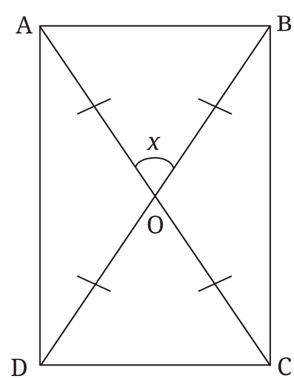
? Can we now identify what type of quadrilateral ABCD is?
Notice that its angles all add up to 90° (30° + 60°).

? What can we say about its sides?

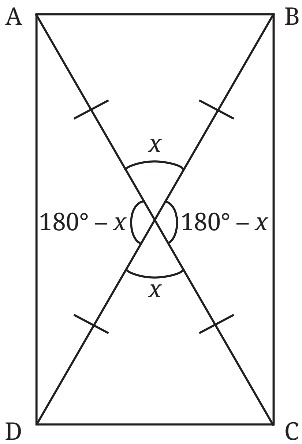


We can see that ΔAOB ≅ ΔCOD and ΔAOD ≅ ΔCOB. Hence, AB = CD, and AD = CB, since they are corresponding parts of congruent triangles.

Therefore, ABCD is a rectangle since it satisfies the definition of a rectangle.

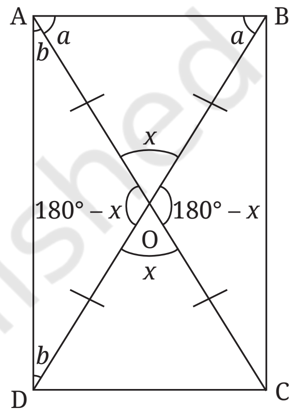


? Will ABCD remain a rectangle if the angles between the diagonals are changed? Can we generalise this?
Take one of the angles between the diagonals as x.



We can compute the four angles between the diagonals to be x, x, $180 - x$, and $180 - x$.

Can you find the other angles?



Since we know that ΔAOB is isosceles, we can denote the measures of both of its base angles by a.

What is the value of a (in degrees) in terms of x?

We have,

$$a + a + x = 180$$

(sum of the interior angles of a triangle)

$$2a = 180 - x$$

$$a = \frac{(180 - x)}{2} = 90 - \frac{x}{2}.$$

Similarly, in the isosceles ΔAOD, let the base angles be b.

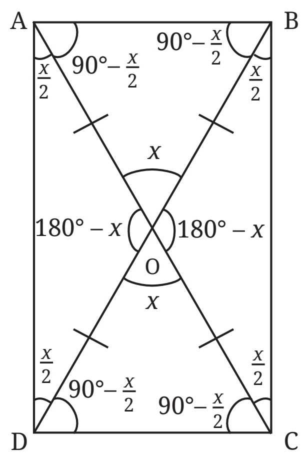
$$b + b + 180 - x = 180$$

$$2b = 180 - (180 - x)$$

$$2b = 180 - 180 + x$$

$$2b = x$$

$$b = \frac{x}{2}.$$



All the angles of the quadrilateral are $a + b$, which is

$$90 - \frac{x}{2} + \frac{x}{2} = 90.$$

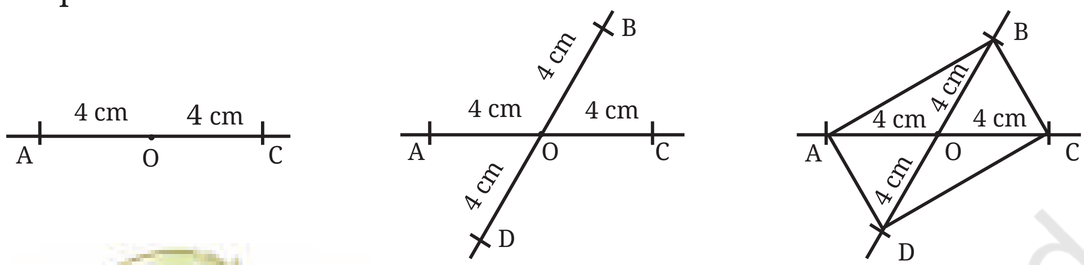
Thus, all four angles of the quadrilateral ABCD are 90°.

What can we say about AB and CD, and AD and BC?

We have ΔAOB ≅ ΔCOD and ΔAOD ≅ ΔCOB. Hence, AB = CD, and AD = CB, since they are the corresponding parts of congruent triangles.

Hence, no matter what the angles between the diagonals are, if the diagonals are equal and they bisect each other, then the angles of the quadrilateral formed are 90° each, and the opposite sides are equal. Thus, the quadrilateral is a rectangle.

Now we know how the wooden strips have to be put together to form the vertices of a rectangle! They should be equal and connected at their midpoints.



This method is actually used in practice to make rectangles. Carpenters in Europe use this method to get a rectangular frame. It is also known that farmers in Mozambique, a country in Africa, use this method while constructing houses to get the base of the house in a rectangular shape.

The Process of Finding Properties

As we have been seeing from lower grades, properties of geometric objects such as parallel lines, angles, and triangles can be deduced through geometric reasoning. We will continue to deduce properties of special types of quadrilaterals in this chapter.

Once you have deduced a property of a quadrilateral, it is good to verify it with a real-world quadrilateral, either the quadrilateral constructed on paper or simply a surface having the shape of the quadrilateral.

If you are not able to figure out the property using deduction, you could experiment by taking real-world quadrilaterals and observing the property through measurement. Note that these observations give useful insights about the property, but with them, we can only form a **conjecture**, that is, **a statement about which we are highly confident, but not yet sure if it always holds true.** For example, constructing a few rectangles and observing through measurement that their diagonals bisect each other does not necessarily mean that this will always be the case—can we be sure that the 1000th rectangle we construct will also have this property? The only way we can be sure of this property is by justifying or proving the statement, just as we did in Deduction 2.

Note to the Teacher: Gently encourage students to deduce or justify properties. Whenever students face challenges in doing it, encourage them to experiment and observe, and use their intuition to figure out the properties.

The Carpenter's Problem shows that rectangles can also be defined as follows—

Rectangle: A rectangle is a quadrilateral whose diagonals are equal and bisect each other.

Observe how different this definition is from the earlier one. Yet, both capture the same class of quadrilaterals. Further, it turns out that the first definition can be simplified.

? In the earlier definition, we stated that a rectangle has (a) opposite sides of equal length, and (b) all angles equal to 90°. Would we be wrong if we just define a rectangle as a quadrilateral in which all the angles are 90°?

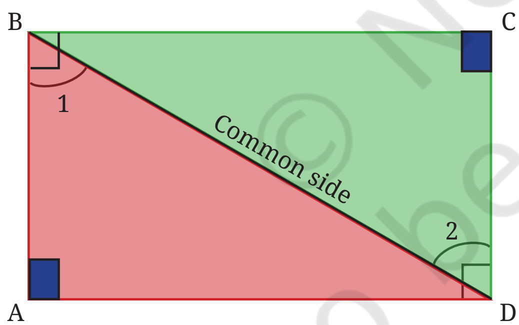
? If you think that this definition is incomplete, try constructing a quadrilateral in which the angles are all 90° but the opposite sides are not equal.

Are you able to construct such a quadrilateral?
Let us prove why this is impossible.

Deduction 4—What is the shape of a quadrilateral with all the angles equal to 90°?

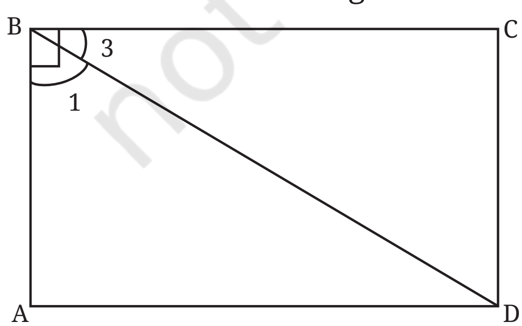
? Consider a quadrilateral ABCD with all angles measuring 90°. What can we say about the opposite sides of such a quadrilateral?

Join BD. ΔBAD and ΔDCB seem congruent. Can we justify this claim?

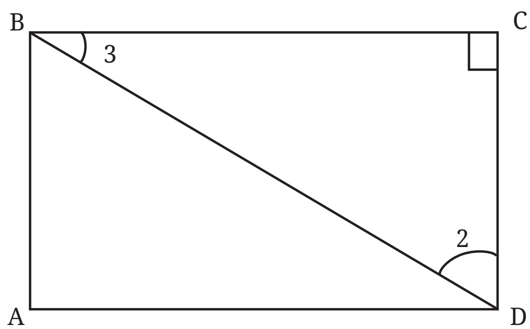


Two equalities can be directly seen in the triangles. What can we say about ∠1 and ∠2?

Recall that we tackled a very similar problem in Deduction 2. We can use the same reasoning here.



Since ∠B = 90°,
∠3 + ∠1 = 90°.



In ΔBCD, since
∠3 + ∠2 + 90° = 180°,
∠3 + ∠2 = 90°.

So, ∠1 = ∠2.

Thus, by the AAS congruence condition, ΔBAD ≅ ΔDCB.

Therefore, AD = CB, and DC = BA, since these are corresponding sides of congruent triangles.

Is it wrong to write ΔBAD ≅ ΔCDB? Why?

Thus, we have established that if all the angles of a quadrilateral are right angles, then the opposite sides have equal lengths. Therefore, the quadrilateral is a rectangle. Thus, a rectangle can simply be defined as follows —

Rectangle: A rectangle is a quadrilateral in which the angles are all 90°.

Let us list the properties of a rectangle.

Property 1: All the angles of a rectangle are 90°.

Property 2: The opposite sides of a rectangle are equal.

Are the opposite sides of a rectangle parallel?

They definitely seem so. This fact can be justified using one of the transversal properties.

Notice that AB acts as a transversal to AD and BC, and that **∠A + ∠B = 90° + 90° = 180°.**

When the sum of the internal angles on the same side of the transversal is 180°, the lines are parallel. We can use this fact to conclude that the lines AD and BC are parallel, which we represent as AD | | BC.

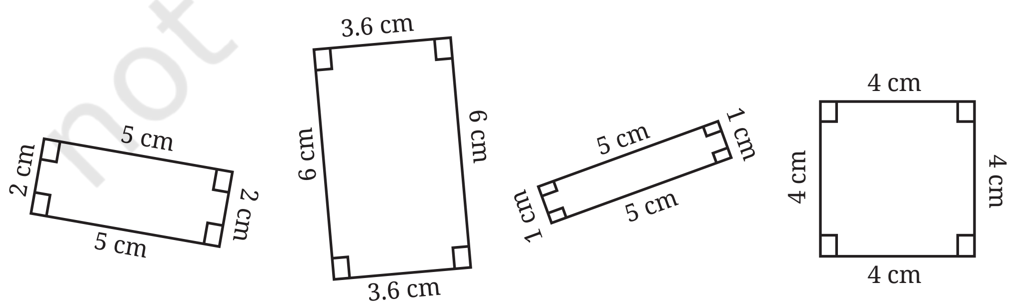
Can you similarly show that AB is parallel to DC (AB | | DC)?

Property 3: The opposite sides of a rectangle are parallel to each other.

Property 4: The diagonals of a rectangle are of equal length and they bisect each other.

A Special Rectangle

In the quadrilaterals below, are there any non-rectangles?

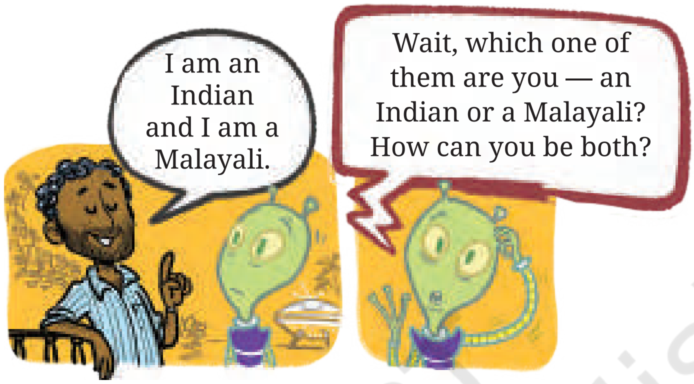


(i) (ii) (iii) (iv)

All these quadrilaterals are rectangles, including (iv). Quadrilateral (iv) is a rectangle because all its angles are 90°. However, it is a special kind of rectangle with all sides of equal length. We know that this quadrilateral is also called a square.

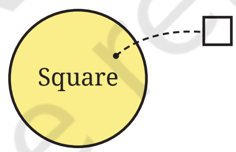
Square: A square is a quadrilateral in which all the angles are equal to 90°, and all the sides are of equal length.

Thus, every square is also a rectangle, but clearly every rectangle is not a square.



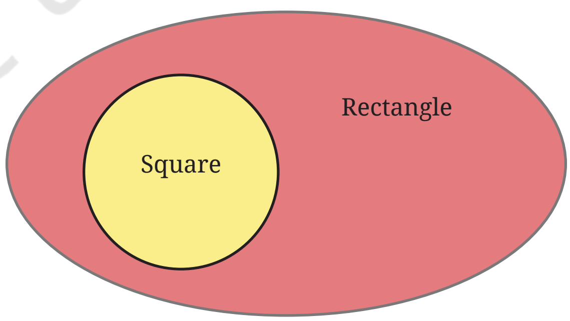
This relation can be pictorially represented using a **Venn diagram**. We have seen these diagrams before. In a Venn diagram, a set of objects is represented as points inside a closed curve. Typically, these closed curves are ovals or circles.

For example, the set of all squares is represented as

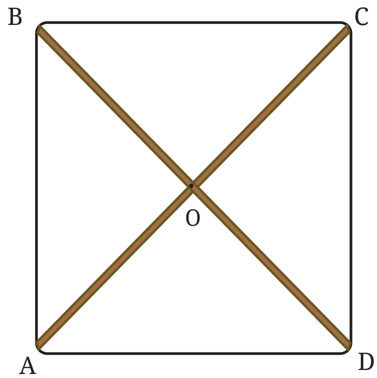


Each point in the region represents a square, thereby covering all the possible squares.

Since every square is a rectangle, the Venn diagram representation of these two sets would be as follows —



Let us consider the Carpenter's Problem again. If the wooden strips have to be placed such that the thread passing through their endpoints forms a square, what must be done?



As in the previous case, let us try to construct a square, one of whose diagonals is of length 8 cm.

While solving the Carpenter's Problem for the case of a rectangle, we have seen that to get a quadrilateral with all angles 90° (and opposite sides of equal length), the diagonals have to be drawn such that —

(i) they are of equal lengths, and

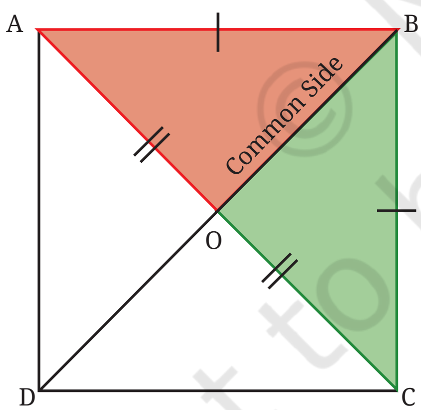
(ii) they bisect each other.

What more needs to be done to get equal sidelengths as well? Can this be achieved by properly choosing the angle between the diagonals? See if you can reason and/or experiment to figure this out!

Deduction 5 — What should be the angle formed by the diagonals?

The angle between the diagonals can be found using the notion of congruence! Suppose we join the equal diagonals such that they bisect each other and result in a square. Let us label the square ABCD.

To find the angle formed by the diagonals, what are the two triangles we should consider for congruence?



By the SSS condition for congruence, $\Delta BOA \cong \Delta BOC$

Can this be used to find the angles $\angle$**BOA and** $\angle$**BOC** formed by the diagonals?

Since these angles are corresponding parts of congruent triangles, they are equal. Further, these angles together form a straight angle. So $\angle$**BOA** + $\angle$**BOC** = **180°.** Thus, these angles have to be 90° each.

This shows that the diagonals of a square bisect each other at right angles. This means that the diagonals have to be drawn such that they are of equal lengths and bisect each other at right angles. Since the endpoints of the diagonals uniquely determine the vertices of a quadrilateral, we will get a square when the diagonals are joined this way.

? Using this fact, construct a square with a diagonal of length 8 cm.

Properties of a Square

Since a square is a special type of rectangle, all the properties of a rectangle hold true for a square.

? Verify if this is true by going through geometric reasoning in Deduction 1 and Deduction 2, and see if they apply to a square as well.

Property 1: All the sides of a square are equal to each other.

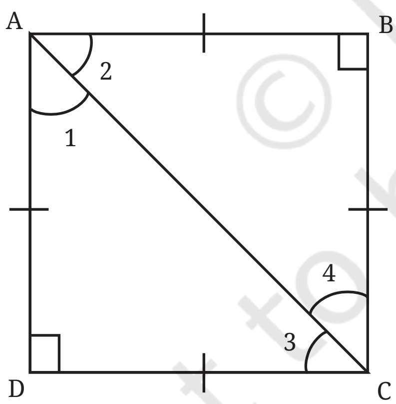
Property 2: The opposite sides of a square are parallel to each other.

Property 3: The angles of a square are all 90°.

Property 4: The diagonals of a square are of equal length and they bisect each other at 90°.

There is one more special property of a square.

? What are the measures of $\angle 1$, $\angle 2$, $\angle 3$, and $\angle 4$? See if you can reason and/or experiment to figure this out!



In ΔADC, we have,
$\angle 1 + \angle 3 + 90 = 180$
Since AD = DC, we have $\angle 1 = \angle 3$.
Thus, $\angle 1 = \angle 3 = 45°$.

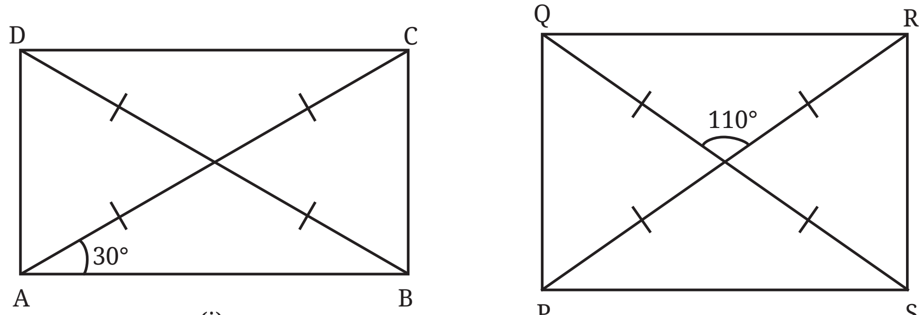
Similarly, find $\angle 2$ and $\angle 4$.

Thus, we have another property of a square —

Property 5: The diagonals of a square divide the angles of the square into equal halves. This can also be expressed as—The diagonals of a square bisect the angles of the square.

Figure it Out

1. Find all the other angles inside the following rectangles.



(i) (ii)

2. Draw a quadrilateral whose diagonals have equal lengths of 8 cm that bisect each other, and intersect at an angle of

   (i) 30° (ii) 40° (iii) 90° (iv) 140°

3. Consider a circle with centre O. Line segments PL and AM are two perpendicular diameters of the circle. What is the figure APML? Reason and/or experiment to figure this out.

4. We have seen how to get 90° using paper folding. Now, suppose we do not have any paper but two sticks of equal length, and a thread. How do we make an exact 90° using these?

   Math Talk

5. We saw that one of the properties of a rectangle is that its opposite sides are parallel. Can this be chosen as a definition of a rectangle? In other words, is every quadrilateral that has opposite sides parallel and equal, a rectangle?

   Try This

4.2 Angles in a Quadrilateral

Is it possible to construct a quadrilateral with three angles equal to 90° and the fourth angle not equal to 90°?

You might have observed through constructions that this may not be possible.

But why not?

This is due to a general property of quadrilaterals related to their angles.

We have seen that the sum of the angles of a triangle is 180°. There is a similar regularity in the sum of the angles of a quadrilateral.

Consider a quadrilateral SOME.

Draw a diagonal SM. We get two triangles ΔSEM and ΔSOM.

In ΔSEM, we have ∠1 + ∠2 + ∠3 = 180°.

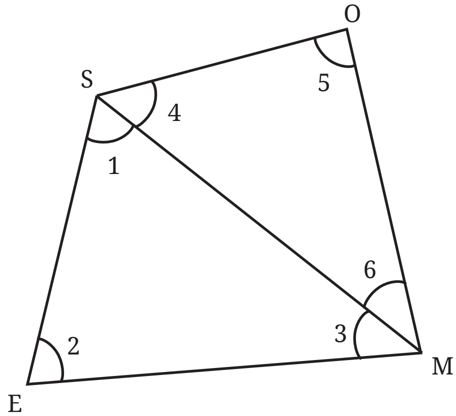
And in ΔSOM, ∠4 + ∠5 + ∠6 = 180°.

What do we get when we add all six angles?

We will have

∠1+ ∠2 + ∠3 + ∠4 + ∠5 + ∠6 = 180° + 180°
= 360°.

Or, (∠1+ ∠4) + (∠3 + ∠6) +∠2 + ∠5 = 360°.



Since (∠1 + ∠4), (∠3 + ∠6), ∠2 and ∠5 are the angles of this quadrilateral, we have the following result—

The sum of all angles in any quadrilateral is 360°.

This explains why it is impossible for a quadrilateral to have three right angles, with the fourth angle not right angle.

4.3 More Quadrilaterals with Parallel Opposite Sides

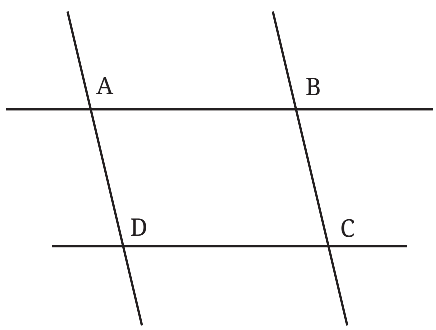
Rectangles (and therefore squares) have parallel opposite sides.

? Are there quadrilaterals that have parallel opposite sides that are not rectangles?

Let us try constructing one.

This can be easily done by drawing two pairs of parallel lines, ensuring that they do not meet at right angles.

? Construct such a figure by recalling how parallel lines can be constructed using a ruler and a set-square, or a compass and a ruler.



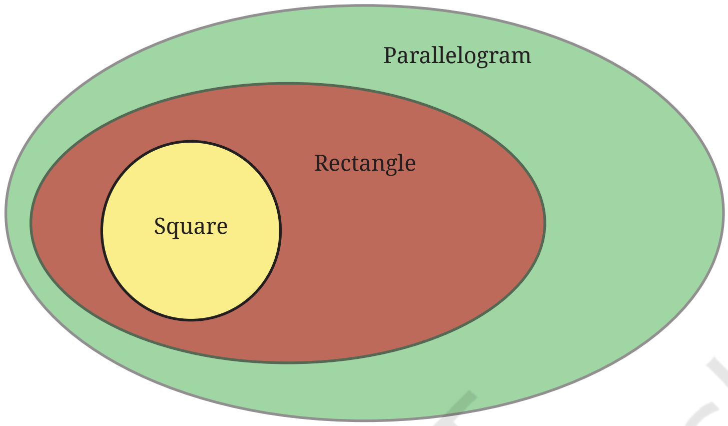
Observe the quadrilateral ABCD. It has parallel opposite sides but is not a rectangle.

Thus, a larger set of quadrilaterals exists in which the opposite sides are parallel. Such quadrilaterals are called **parallelograms.**

? Is a rectangle a parallelogram?

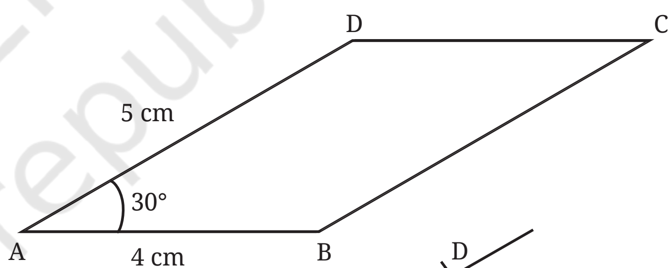
A rectangle has opposite sides parallel. So, it satisfies the parallelogram's definition. Hence, it is indeed a parallelogram. More specifically, a rectangle is a special kind of parallelogram with all its angles equal to 90°.

Let us represent this relation using a Venn diagram.

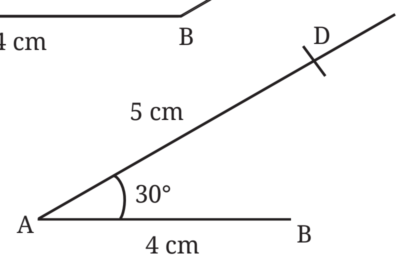


To understand the relations between the sides and the angles of a parallelogram, let us construct the following figure.

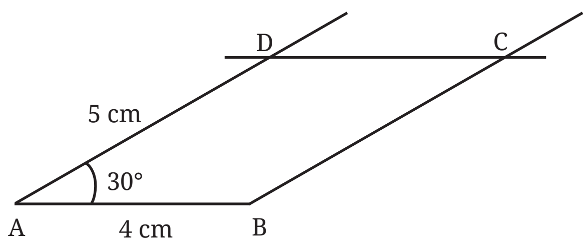
? Draw a parallelogram with adjacent sides of lengths 4 cm and 5 cm, and an angle of 30° between them.



Step 1: Draw line segments AB = 4 cm and AD = 5 cm with an angle of 30° between them.



Step 2: Draw a line parallel to AB through the point D, and a line parallel to AD through B. Mark the point at which these lines intersect as C.

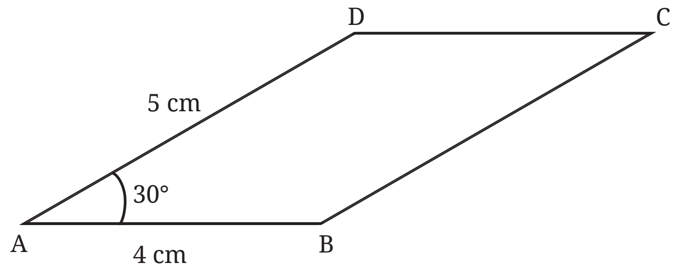


ABCD is the required parallelogram.

? What are the remaining angles of the parallelogram? What are the lengths of the remaining sides? See if you can reason out and/or experiment to figure these out.

Deduction 6— What can we say about the angles of a parallelogram?

In the parallelogram ABCD, AB||CD, and AD is a transversal to them.



∠A + ∠D = 180° (sum of the internal angles on the same side of a transversal).

Therefore,
∠D = 180 – ∠A = 180 – 30 = 150°.

Similarly, AD||BC, and AB and CD are transversals to them.

So, ∠A + ∠B = 180°.

So, ∠C + ∠D = 180°.

Using these equations, we get ∠B = 150° and ∠C = 30°.

We see that in this parallelogram, the **adjacent pairs of angles add up to 180° and opposite pairs of angles are equal.**

Thus,

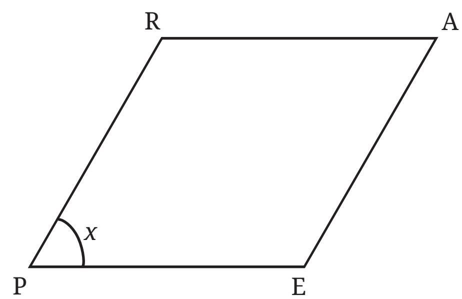
∠A + ∠B = 180°, ∠A + ∠D = 180°, ∠C + ∠D = 180°, and ∠B + ∠C = 180°.

And,

∠A = ∠C, and ∠B = ∠D.

Since the adjacent angles are the interior angles on the same side of a transversal to a pair of parallel lines, they must add up to 180°.

What about the opposite angles? Will they be equal in all parallelograms? If yes, how can we be sure?



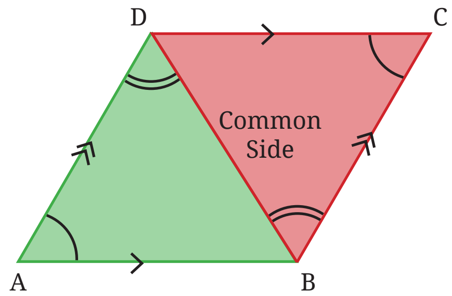
Let us take one of the angles to be x.
What are the other angles?
Since ∠P + ∠R = 180°,
∠R = 180 – ∠P = 180 – x.
Similarly, since ∠A + ∠R = 180°,
∠A = 180 – ∠R = 180 – (180 – x) = 180 – 180 + x = x.
Thus, ∠P = ∠A = x.
Similarly, we can deduce that ∠ R = ∠E = 180 – x.

Therefore, this shows that **the opposite angles of a parallelogram are always equal.**

Deduction 7— What can we say about the sides of a parallelogram?

By looking at a parallelogram, it appears that the opposite sides are equal.

Can we again use congruence to show this? Which two triangles can be considered for this?



In ΔABD and ΔCDB, the angles marked with a single arc are equal as they are the opposite angles of a parallelogram.

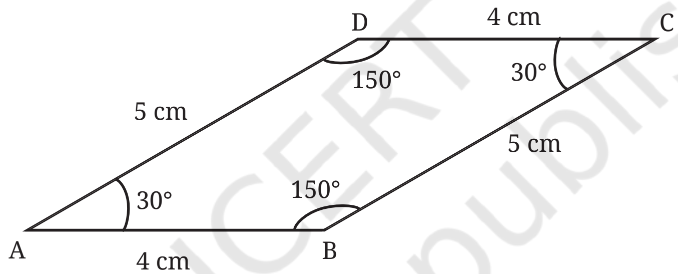
Since AD | | BC, and BD is a transversal to it, the angles marked with double arcs are equal as they are alternate angles.

So, by the AAS condition, the triangles are congruent, that is, ΔABD ≅ ΔCDB.

Therefore, AD = CB, and AB = CD.
Thus, **the opposite sides of a parallelogram are equal.**

? Is it wrong to write ΔABD ≅ ΔCBD? Why?

From these deductions we can find the remaining sides and angles of the parallelogram.



Let us list the properties of a parallelogram.

Property 1: The opposite sides of a parallelogram are equal.

Property 2: The opposite sides of a parallelogram are parallel.

Property 3: In a parallelogram, the adjacent angles add up to 180°, and the opposite angles are equal.

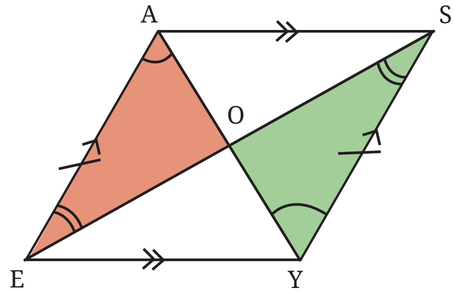
? Are the diagonals of a parallelogram always equal? Check with the parallelogram that you have constructed.

We see that the diagonals of a parallelogram need not be equal.

? Do they bisect each other (do they intersect at their midpoints)? Reason and/or experiment to figure this out.

Deduction 8— What is the point of intersection of the two diagonals in a parallelogram?

As in the case of a rectangle, we can find out if the diagonals bisect each other by examining the congruence of ΔAOE and ΔYOS in the parallelogram EASY.



AE = YS (as they are the opposite sides of the parallelogram)

The angles marked using a single arc are equal, and so are the angles marked using a double arc, since they are alternate angles of parallel lines.

Thus, by the ASA condition, the triangles are congruent, that is,
$\Delta AOE \cong \Delta YOS$.

Therefore, OA = OY, and OE = OS, since they are corresponding parts of congruent triangles.
Thus, O is the midpoint of both diagonals.

? Is it wrong to write $\Delta AOE \cong \Delta SOY$? Why?

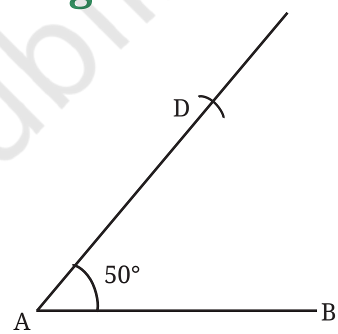
Property 4: The diagonals of a parallelogram bisect each other.

? Do the diagonals of a parallelogram intersect at a particular angle?

4.4 Quadrilaterals with Equal Sidelengths

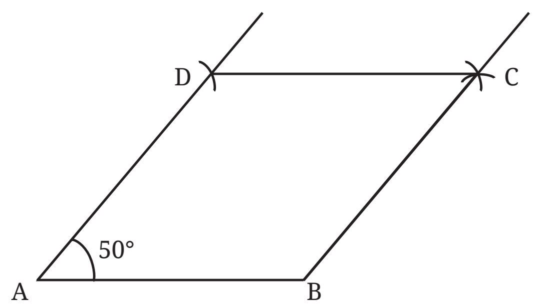
? Are squares the only quadrilaterals that have equal sidelengths? Let us explore this question through construction.

Draw two equal sides AD and AB, that are not perpendicular to each other.



? Can we complete this quadrilateral so that all its sides are of the same length?

Mark a point C whose distance from B and D is equal to AB (or AD). To do this, measure AB using a compass. Keeping this length as the radius, cut arcs from B and D.



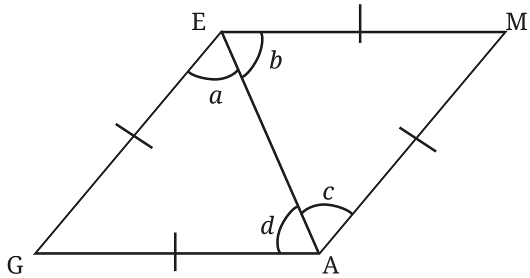
Now we have a quadrilateral with equal sidelengths and one of its angles 50°. Note that we could have constructed such a quadrilateral by taking any angle less than 180° (in place of 50°).

A quadrilateral in which all the sides have the same length is a rhombus.

? What are the other angles of the rhombus ABCD that we have constructed? Reason and/or experiment to figure this out.

Deduction 9— What can we say about the angles in a rhombus?

Consider a rhombus GAME.
In ΔGAE, since GE = GA, $a = d$.
Similarly, in ΔMAE, since ME = MA, $b = c$.



? It can be seen that ΔGAE ≅ ΔMAE (How?)

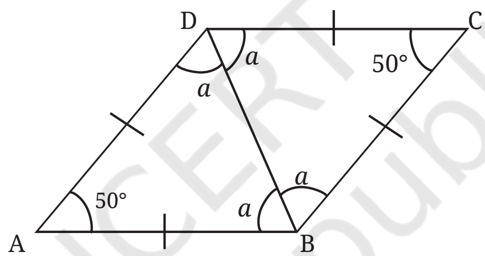
So, $a = b$, $c = d$ and ∠G = ∠M (since they are corresponding parts of congruent triangles).
Thus, we have, $a = b = c = d$.

These facts hold for any rhombus. Let us apply them to the rhombus ABCD that we constructed earlier. Let the four equal angles formed by the diagonal be a, as shown in the figure

In ΔADB, we have

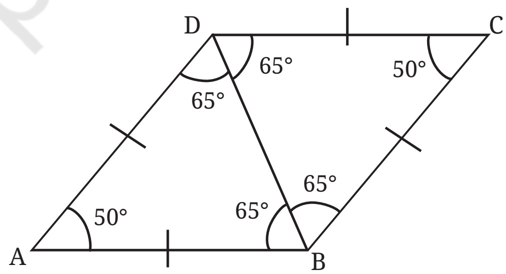
$a + a + 50 = 180°$.

So, $a = 65°$.

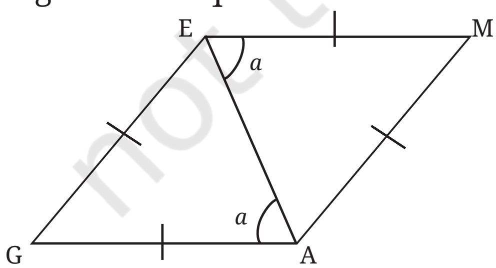


Thus, the angles of the rhombus *ABCD* are 50°, 130°, 50°, and 130°.

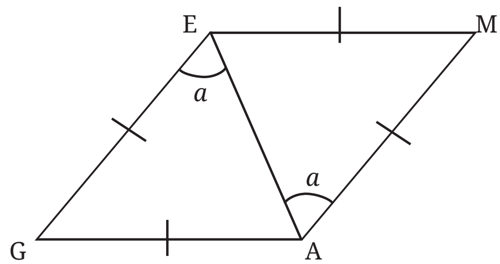
So, **in a rhombus opposite angles are equal to each other.**



Interestingly, there is one more way by which we could have figured out the other angles of the rhombus ABCD. We have shown that in a general rhombus GAME, the four angles formed by a diagonal are equal to each other.

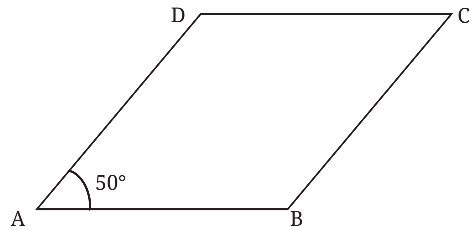


Consider the lines EM and GA and its transversal AE. Since the alternate angles are equal, EM||GA.



Similarly consider the lines GE and AM and its transversal AE. Since the alternate angles are equal, GE||AM.

As opposite sides are parallel, GAME is also a parallelogram. Thus, every rhombus is a parallelogram, and the properties of a parallelogram hold true for a rhombus as well. Thus, the adjacent angles of a rhombus add up to 180°, and the opposite angles are equal (verify that the arguments in Deduction 6 can be applied to a rhombus as well!).



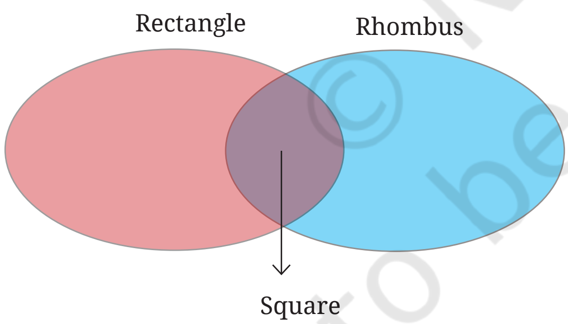
Thus, in rhombus ABCD,

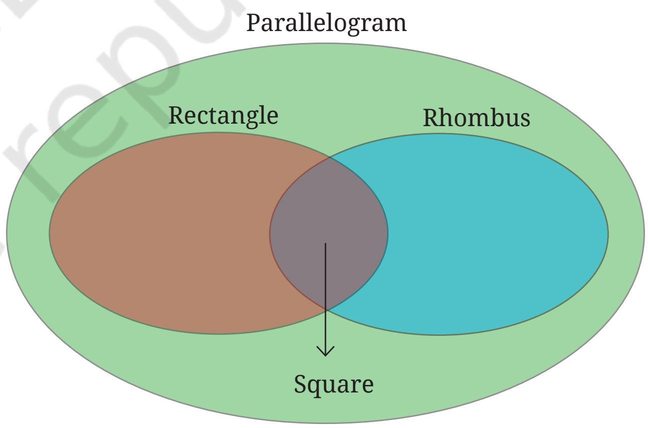
$$\angle A = \angle C = 50°, \text{ and}$$
$$\angle D = \angle B = 180 - 50 = 130°.$$

? So a rhombus is a parallelogram, and a rectangle is also a parallelogram. How can this be represented using a Venn diagram?

? Where will the set of squares occur in this diagram?

We know that a square is a rectangle. Since the opposite sides of a square are parallel, a square is also a parallelogram. Further, since all the sides of a square have the same length, a square is also a rhombus. Thus, the Venn diagram will be as follows.





Let us list the properties of a rhombus.

Property 1: All the sides of a rhombus are equal to each other.

Property 2: The opposite sides of a rhombus are parallel to each other.

Property 3: In a rhombus, the adjacent angles add up to 180°, and the opposite angles are equal.

? Are the diagonals of a rhombus equal?

Property 4: The diagonals of a rhombus bisect each other.

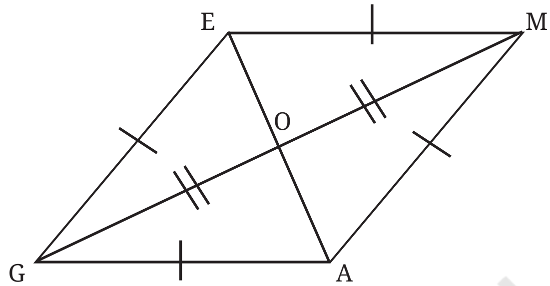
Property 5: The diagonals of a rhombus bisect its angles.

? Do the diagonals of a rhombus intersect at any particular angle? Reason out and/or experiment to figure this out!

Deduction 10—What can we say about the angles formed by the diagonals of a rhombus at their point of intersection?

? In the rhombus GAME, we have $\Delta GEO \cong \Delta MEO$ (why?).

So, $\angle GOE = \angle MOE$, as they are corresponding parts of congruent triangles. As they add up to 180°, they should be 90° each.

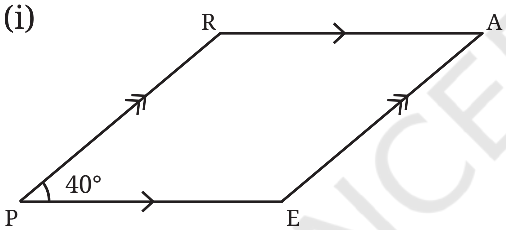


Property 6: Diagonals of a rhombus intersect each other at an angle of 90°.

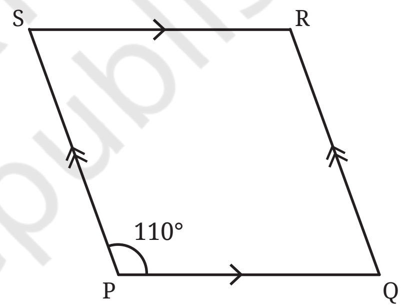
? Figure it Out

1. Find the remaining angles in the following quadrilaterals.

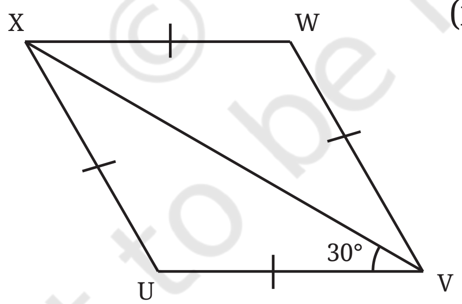
(i)



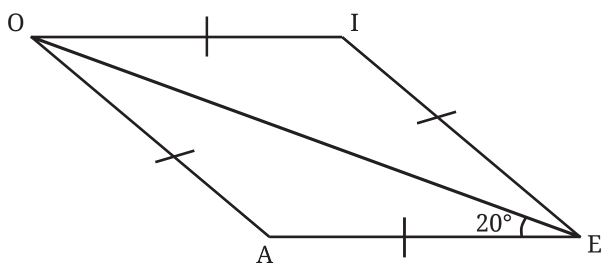
(ii)



(iii)



(iv)



2. Using the diagonal properties, construct a parallelogram whose diagonals are of lengths 7 cm and 5 cm, and intersect at an angle of 140°.

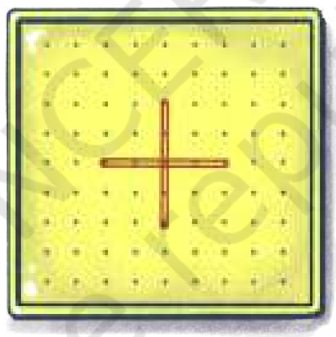
3. Using the diagonal properties, construct a rhombus whose diagonals are of lengths 4 cm and 5 cm.

4.5 Playing with Quadrilaterals

Geoboard Activity

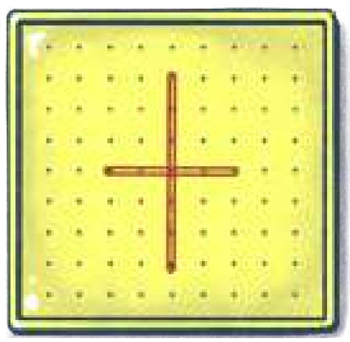
Take a geoboard and some rubber bands. If you do not have these, you could just use the dot grid papers given at the end of the book for this activity.

Place two rubber bands perpendicular to each other, forming diagonals of equal length. Join the ends.



? What is the quadrilateral that you get? Justify your answer.

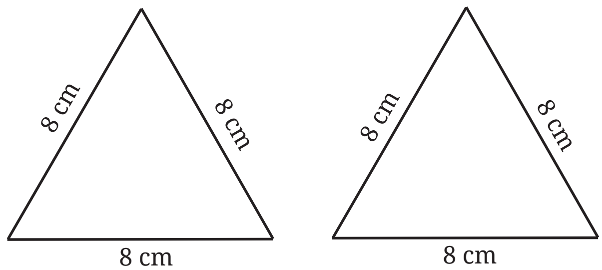
Extend one of the diagonals on both sides by 2 cm.



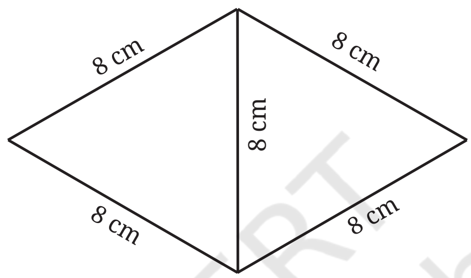
? What quadrilateral will you get now? Justify your answer.

Joining Triangles

1. Take two cardboard cutouts of an equilateral triangle of sidelength 8 cm.

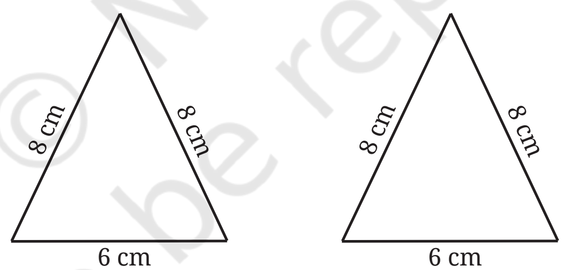


? Can you join them to get a quadrilateral?



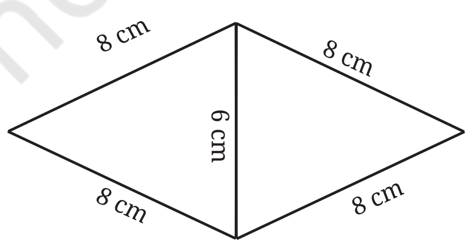
? What type of a quadrilateral is this? Justify your answer.

2. Take two cardboard cutouts of an isosceles triangle with sidelengths 8 cm, 8 cm, and 6 cm.

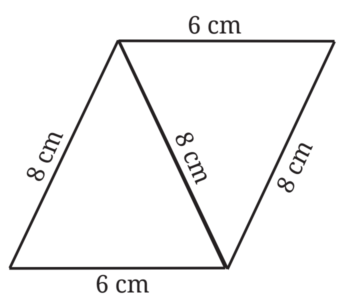


? What are the different ways they can be joined to get a quadrilateral?

Joining them in this way you get

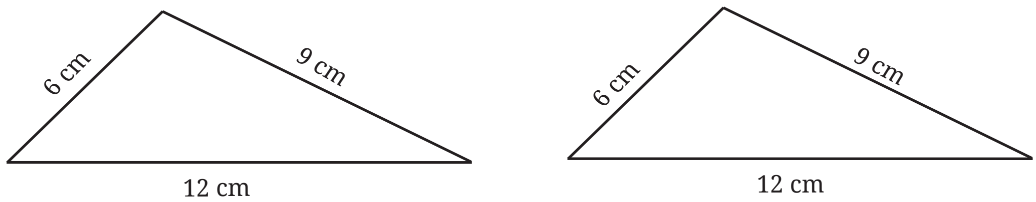


Joining them in this way you get



? What quadrilaterals are these? Justify your answers.

3. Take two cardboard cutouts of a scalene triangle with sides 6 cm, 9 cm, and 12 cm.



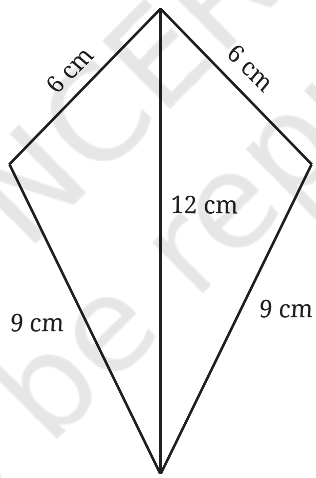
? What are the different ways they can be joined to get a quadrilateral?

? Are you able to identify the different quadrilaterals that are obtained by joining the triangles? Justify your answer whenever you identify a quadrilateral.

4.6 Kite and Trapezium

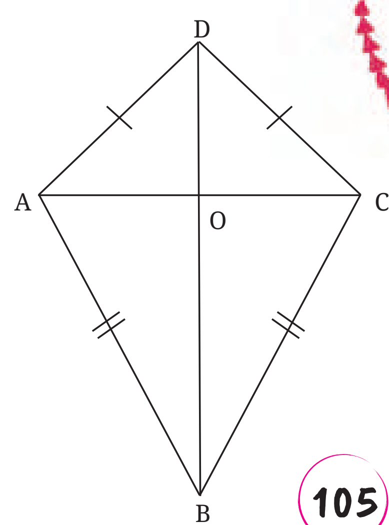
Kite

One of the ways the two triangles of sides 6 cm, 9 cm and 12 cm can be joined together is as follows —



This quadrilateral looks like a kite. Observe that the adjacent sides are of the same length.

Kite: A kite is a quadrilateral that can be labelled ABCD such that AB = BC, and CD = DA.



? **Property 1:** In the kite, show that the diagonal BD

(i) bisects ∠ABC and ∠ADC,

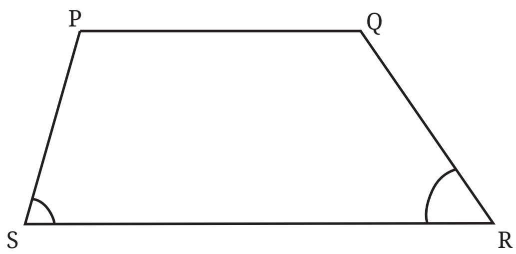
(ii) bisects the diagonal AC, that is, AO = OC, and is perpendicular to it.

Hint: Is ΔAOB ≅ ΔCOB?

Trapezium

Parallelograms are quadrilaterals that have parallel opposite sides. We get a new type of quadrilateral if we relax this condition.

Trapezium: A trapezium is a quadrilateral with at least one pair of parallel opposite sides.



? Construct a trapezium. Measure the base angles (marked in the figure).

? Can you find the remaining angles without measuring them?

Since PQ | | SR, we have

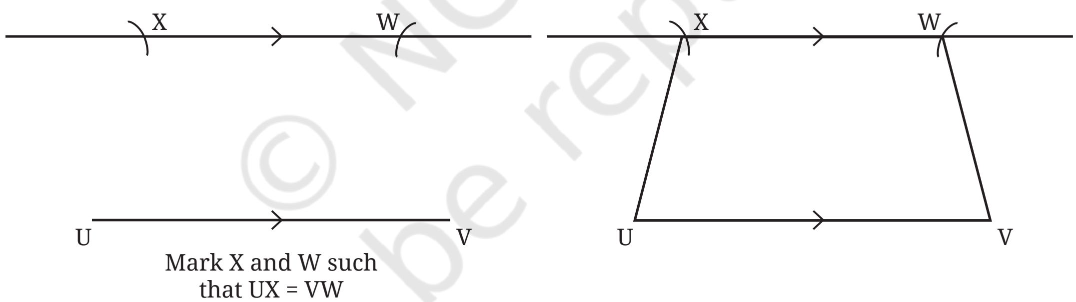
Property 1: ∠S + ∠P = **180°** and ∠R + ∠Q = **180°**.

Using these facts, the remaining angles can easily be found. Verify your answer after finding them.

When the non-parallel sides of a trapezium have the same lengths, the trapezium is called an **isosceles trapezium.**

? How do we construct an isosceles trapezium?

? Construct an isosceles trapezium UVWX, with UV | | XW. Measure ∠U.



Can you find the remaining angles without measuring them?

Does it appear that the angles opposite to the equal sides—∠U and ∠V—are also equal? Can we find congruent triangles here? Consider line segments XY and WZ perpendicular to UV.

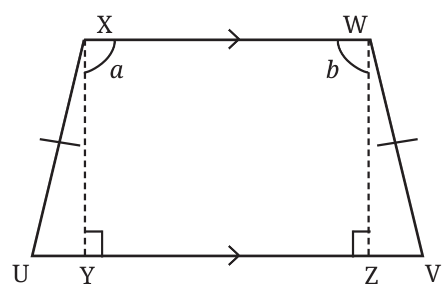
? What type of quadrilateral is XWZY?

Since XW | | UV,

a = 180° – ∠XYZ = 90°, and

b = 180° – ∠WZY = 90° (since the internal angles on the same side of a transversal add up to 180°)

Hence, XWZY is a rectangle.



Now, it can be shown that ΔUXY ≅ ΔVWZ. (How?)

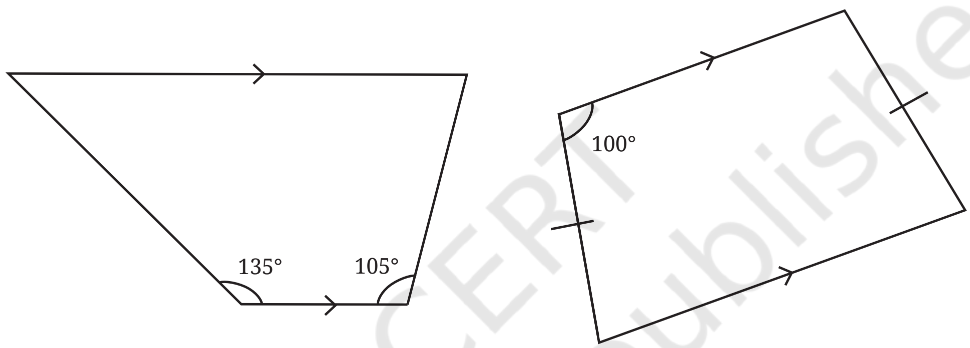
Thus, ∠U = ∠V.

Using this fact, the remaining angles of the isosceles trapezium can be determined. Verify the angles by measurement.

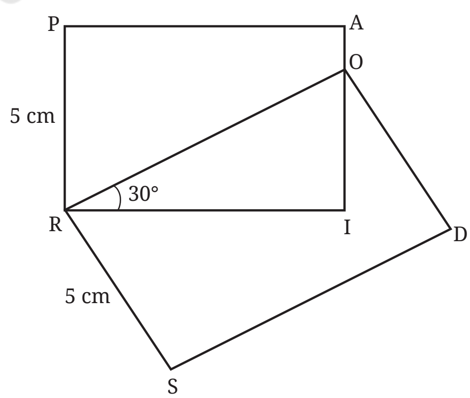
Property 2: In an isosceles trapezium, the angles opposite to the equal sides are equal.

Figure it Out

1. Find all the sides and the angles of the quadrilateral obtained by joining two equilateral triangles with sides 4 cm.
2. Construct a kite whose diagonals are of lengths 6 cm and 8 cm.
3. Find the remaining angles in the following trapeziums—

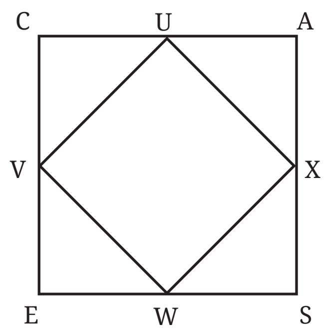


4. Draw a Venn diagram showing the set of parallelograms, kites, rhombuses, rectangles, and squares. Then, answer the following questions—
   (i) What is the quadrilateral that is both a kite and a parallelogram?
   (ii) Can there be a quadrilateral that is both a kite and a rectangle?
   (iii) Is every kite a rhombus? If not, what is the correct relationship between these two types of quadrilaterals?
5. If PAIR and RODS are two rectangles, find ∠IOD.

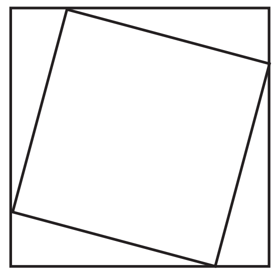


6. Construct a square with diagonal 6 cm without using a protractor.

7. CASE is a square. The points U, V, W and X are the midpoints of the sides of the square. What type of quadrilateral is UVWX? Find this by using geometric reasoning, as well as by construction and measurement. Find other ways of constructing a square within a square such that the vertices of the inner square lie on the sides of the outer square, as shown in Figure (b).



(a)



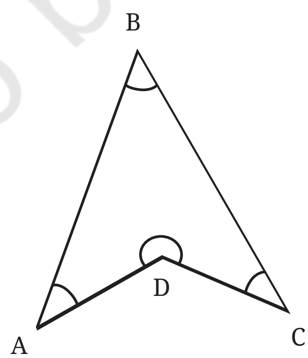
(b)

8. If a quadrilateral has four equal sides and one angle of 90°, will it be a square? Find the answer using geometric reasoning as well as by construction and measurement.

9. What type of a quadrilateral is one in which the opposite sides are equal? Justify your answer.

   Hint: Draw a diagonal and check for congruent triangles.

10. Will the sum of the angles in a quadrilateral such as the following one also be 360°? Find the answer using geometric reasoning as well as by constructing this figure and measuring.



11. State whether the following statements are true or false. Justify your answers.

    (i) A quadrilateral whose diagonals are equal and bisect each other must be a square.

(ii) A quadrilateral having three right angles must be a rectangle.

(iii) A quadrilateral whose diagonals bisect each other must be a parallelogram.

(iv) A quadrilateral whose diagonals are perpendicular to each other must be a rhombus.

(v) A quadrilateral in which the opposite angles are equal must be a parallelogram.

(vi) A quadrilateral in which all the angles are equal is a rectangle.

(vii) Isosceles trapeziums are parallelograms.

SUMMARY

- A **rectangle** is a quadrilateral in which the angles are all 90°.

  Properties of a rectangle —

  - Opposite sides of a rectangle are equal.
  - Opposite sides of a rectangle are parallel to each other.
  - Diagonals of a rectangle are of equal length and they bisect each other.

- A **square** is a quadrilateral in which all the angles are 90°, and all the sides are of equal length.

  Properties of a square —

  - The opposite sides of a square are parallel to each other.
  - The diagonals of a square are of equal lengths and they bisect each other at 90°.
  - The diagonals of a square bisect the angles of the square.

- A **parallelogram** is a quadrilateral in which opposite sides are parallel.

  Properties of a parallelogram —

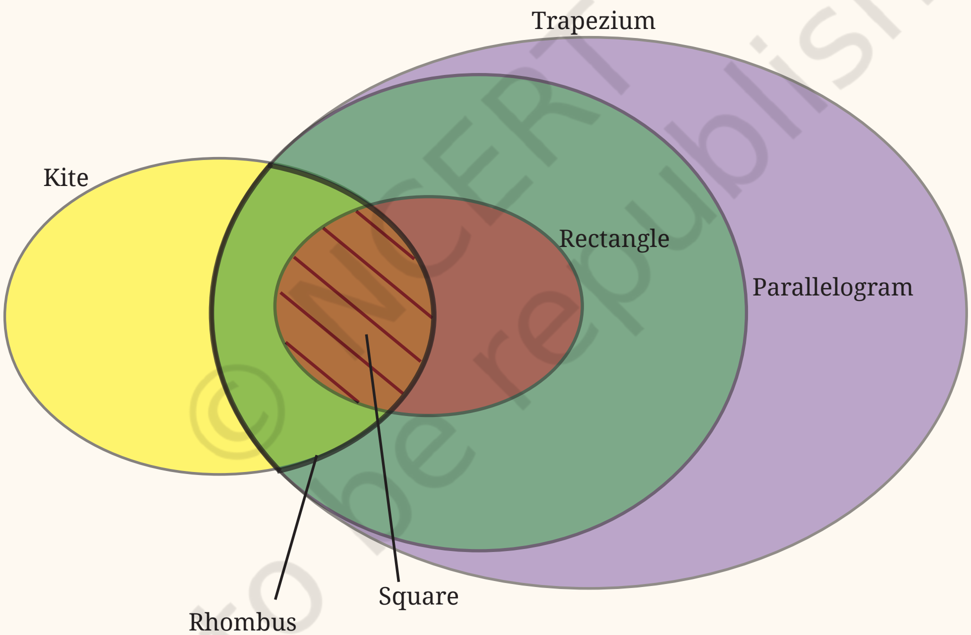
  - The opposite sides of a parallelogram are equal.
  - In a parallelogram, the adjacent angles add up to 180°, and the opposite angles are equal.
  - The diagonals of a parallelogram bisect each other.

- A **rhombus** is a quadrilateral in which all the sides have the same length.

Properties of a rhombus —

- The opposite sides of a rhombus are parallel to each other.
- In a rhombus, the adjacent angles add up to 180°, and the opposite angles are equal.
- The diagonals of a rhombus bisect each other at right angles.
- The diagonals of a rhombus bisect its angles.

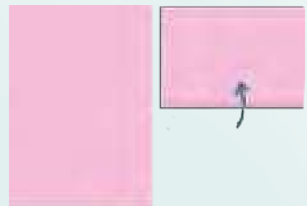
- A **kite** is a quadrilateral with two non-overlapping adjacent pairs of sides having the same length.
- A **trapezium** is a quadrilateral having at least one pair of parallel opposite sides.
- The sum of the angle measures in a quadrilateral is 360°.



IT'S PUZZLE TIME!

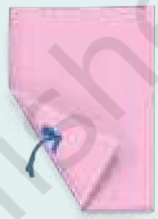
Which Quad?

Gameplay

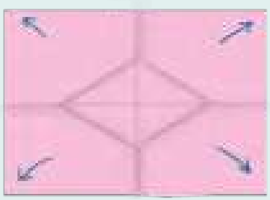


1. Fold a sheet into half.

2. Now, fold it once more into a quarter.



3. Make a triangular crease at the corner that is at the middle of the paper.



4. Open the sheet. What is the shape formed by the creases?

5. How would you fold the quarter paper to get the kinds of creases shown in the following image.

6. How would you fold the quarter paper such that a square is formed?

5 NUMBER PLAY

5.1 Is This a Multiple Of?

Sum of Consecutive Numbers

Anshu is exploring sums of consecutive numbers. He has written the following—

$$
\begin{aligned}
7 &= 3 + 4\\
10 &= 1 + 2 + 3 + 4\\
12 &= 3 + 4 + 5\\
15 &= 7 + 8\\
&= 4 + 5 + 6\\
&= 1 + 2 + 3 + 4 + 5
\end{aligned}
$$

Now, he is wondering—

- "Can I write every natural number as a sum of consecutive numbers?"
- "Which numbers can I write as the sum of consecutive numbers in more than one way?"
- "Ohh, I know all odd numbers can be written as a sum of two consecutive numbers. Can we write all even numbers as a sum of consecutive numbers?"
- "Can I write 0 as a sum of consecutive numbers? Maybe I should use negative numbers."

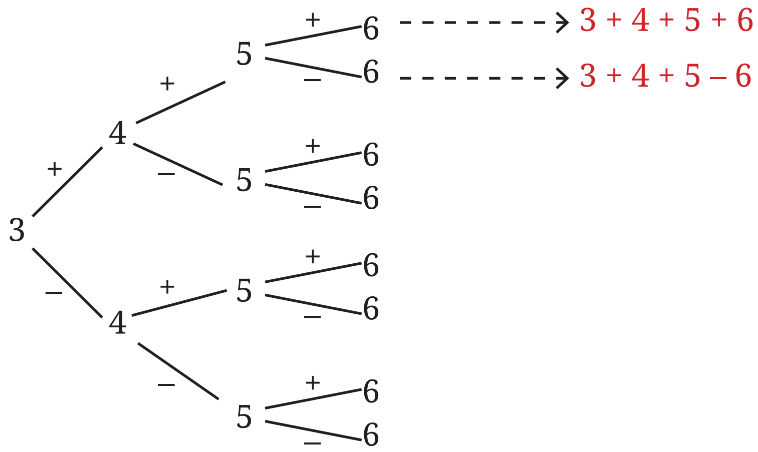
? Explore these questions and any others that may occur to you. Discuss them with the class.

Math Talk

? Take any 4 consecutive numbers. For example, 3, 4, 5, and 6. Place '+' and '–' signs in between the numbers. How many different possibilities exist? Write all of them.

$$3 + 4 - 5 + 6$$
$$3 - 4 - 5 - 6$$

Eight such expressions are possible. You can use the diagram below to systematically list all the possibilities.



? Evaluate each expression and write the result next to it. Do you notice anything interesting?

? Now, take four other consecutive numbers. Place the '+' and '–' signs as you have done before. Find out the results of each expression. What do you observe?

? Repeat this for one more set of 4 consecutive numbers. Share your findings.

Math Talk

3 + 4 – 5 + 6 = 8 3 – 4 – 5 – 6 = – 12 . . .	5 + 6 – 7 + 8 = 12 5 – 6 – 7 – 8 = – 16 . . .	__ + __ – __ + __ = __ __ – __ – __ – __ = __ . . .

Some sums appear always no matter which 4 consecutive numbers are chosen. Isn't that interesting?

? Do these patterns occur no matter which 4 consecutive numbers are chosen? Is there a way to find out through reasoning?

Hint: Use algebra and describe the 8 expressions in a general form.

You might have noticed that the results of all expressions are even numbers. Even numbers have a factor of 2. Negative numbers having a factor 2 are also even numbers, for example, – 2, – 4, – 6, and so on. Check if anyone in your class got an odd number.

When 4 consecutive numbers are chosen, no matter how the '+' and '–' signs are placed between them, the resulting expressions always have even parity.

Now take any 4 numbers, place '+' and '–' signs in the eight different ways, and evaluate the resulting expression. What do you observe about their parities?
Repeat this with other sets of 4 numbers.

? Is there a way to explain why this happens?

Math Talk

Hint: Think of the rules for parity of the sum or difference of two numbers.

Explanation 1: Let us consider any of the 8 expressions formed by four numbers a, b, c, and d. When one of its signs is switched, its value always increases or decreases by an even number! Let us see why.
Consider one of the expressions: $a + b - c - d$.
Replacing $+b$ by $-b$, we get

$$a - b - c - d.$$

By how much has the number changed? It has changed by

$$(a + b - c - d) - (a - b - c - d)$$

$= a + b - c - d - a + b + c + d$ (notice how the signs changed when we opened the second set of brackets)

$= 2b$ (this is an even number).

If the difference between two numbers is even, can they have different parities? No! So either both are even or both are odd.

Now, let us see what happens when a negative sign is switched to a positive sign.

? Replace any negative sign in the expression $a + b - c - d$ with a positive sign and find the difference between the two numbers.

? What do you conclude from this observation?

Starting from any expression, we can get 7 expressions by switching one or more '+' and '–' signs. Thus, all the expressions have the same parity!

Explanation 2: We know that

odd ± odd = even

even ± even = even

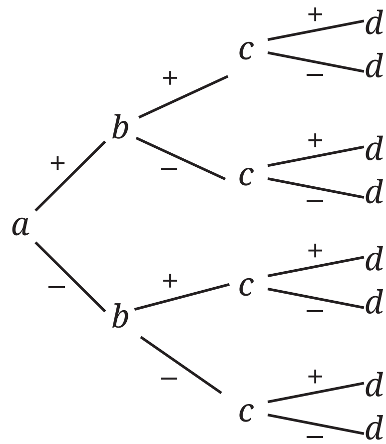
odd ± even = odd.

We have seen that the parity of $a + b$ and $a - b$ is the same, regardless of the parities of a and b.

In short, $a \pm b$ have the same parity. By the same argument, $a \pm b + c$ and $a \pm b - c$ have the same parity. Extending this further, we can say that all the expressions $a \pm b \pm c \pm d$ have the same parity.

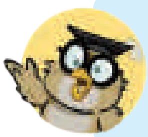
Explanation 3: This can also be explained using the positive and negative token model you studied in the chapter on Integers. Try to think how.

The number of ways to choose 4 numbers a, b, c, d and combine them using '+' and '–' signs is infinite. Mathematical reasoning allows us to prove that all the combinations $a \pm b \pm c \pm d$ always have the same parity, without having to go through them one by one.



Several problems in mathematics can be thought about and solved in different ways. While the method you came up with may be dear to you, it can be amusing and enriching to know how others thought about it. Two tidbits: 'share' and 'listen'.

? Is the phenomenon of all the expressions having the same parity limited to taking 4 numbers? What do you think?



'What if ...?', 'Will it always happen?'— Wondering and posing questions and conjectures is as much a part of mathematics as problem solving.

Breaking Even

We know how to identify even numbers. Without computing them, find out which of the following arithmetic expressions are even.

43 + 37	672 – 348	4 × 347 × 3	708 – 477
809 + 214	119 × 303	543 – 479	513^3

? Using our understanding of how parity behaves under different operations, identify which of the following algebraic expressions give an even number for any integer values for the letter-numbers.

$2a + 2b$	$3g + 5h$	$4m + 2n$	$2u - 4v$	
$13k - 5k$	$6m - 3n$	$x^2 + 2$	$b^2 + 1$	$4k \times 3j$

The expression $4m + 2q$ will always evaluate to an even number for any integer values of m and q. We can justify this in two different ways—

- We know $4m$ is even and $2q$ is even for any integers m and q. Therefore, their sum will also be even.
- The expression $4m + 2q$ is equal to the expression $2(2m + q)$. Here, the expression $2(2m + q)$ means 2 times $2m + q$. In other words, 2 is a factor of this expression. Therefore, this expression will always give an even number for any integers m and q.

For example, if $m = 4$ and $q = -9$, the expression $4m + 2q$ becomes $4 \times 4 + 2 \times (-9) = -2$, which is an even number.

In the expression $x^2 + 2$, x^2 is even if x is even, and x^2 is odd if x is odd. Therefore, the expression $x^2 + 2$ will **not** always give an even number. An example and a non-example for when the expression evaluates to an even number — (i) if $x = 6$, then $x^2 + 2 = 38$, and (ii) if $x = 3$, then $x^2 + 2 = 11$.

? Similarly, determine and explain which of the other expressions always give even numbers. Write a couple of examples and non-examples, as appropriate, for each expression.

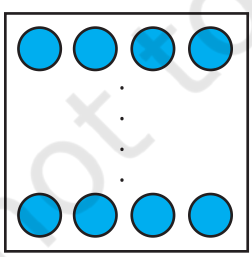
? Write a few algebraic expressions which always give an even number.

Pairs to Make Fours

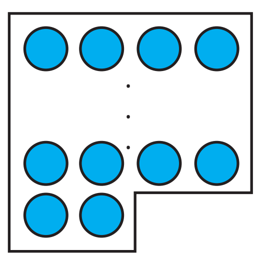
? Take a pair of even numbers. Add them. Is the sum divisible by 4?

Try this with different pairs of even numbers.
When is the sum a multiple of 4, and when is it not?
Is there a general rule or a pattern?

Even numbers can be of two types based on the remainders they leave when divided by 4.



Even numbers that are multiples of 4 leave a remainder of 0 when divided by 4.

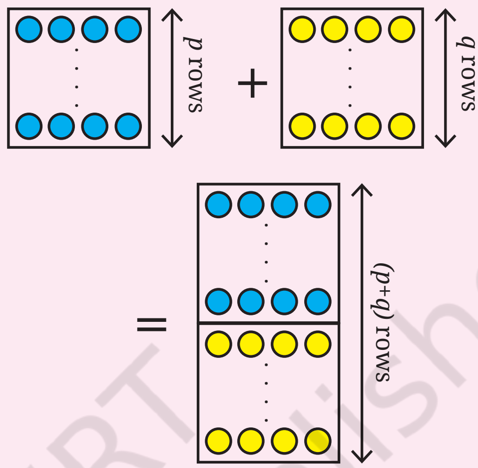
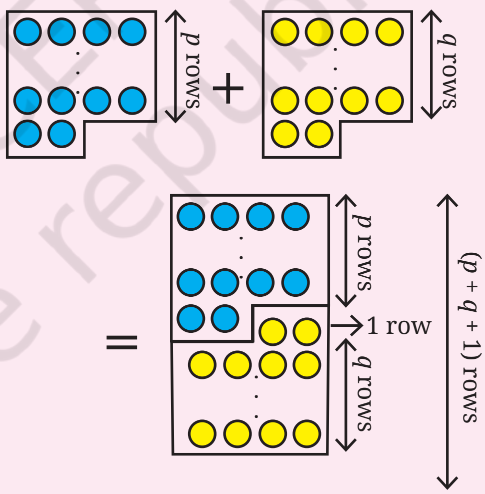


Even numbers that are not multiples of 4 leave a remainder 2 when divided by 4.

? When will two even numbers add up to give a multiple of 4?

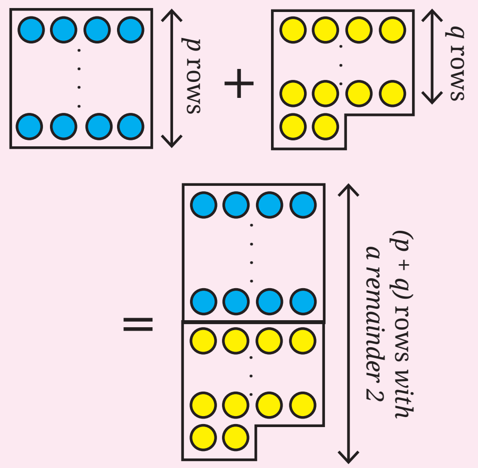
This problem is similar to the question of identifying when adding two numbers will result in an even number. Can you see this?

There are three cases to examine:

Explanation with Algebra and Visualisation			Examples
Adding two (even) numbers that are multiples of 4 will always give a multiple of 4.	$4p$ and $4q$. $4p + 4q$ $= 4(p + q)$.		4, 12, 16, 24, 36. 12 + 16 = 4 (3 + 4) = 28. 16 + 28 = 4 (4 + 7) = 44.
Adding two even numbers that are not multiples of 4 will always give a multiple of 4 because their remainders of 2 add up to 4.	$(4p + 2)$ and $(4q + 2)$. $(4p + 2) + (4q + 2)$ $= 4p + 4q + 4$ $= 4(p + q + 1)$.		2, 6, 10, 18, 22, 42. 2 + 6 = 8. 6 + 10 = 16. 22 + 6 = 28.

What happens when we add a multiple of 4 to an even number that is not a multiple of 4? Is it similar to the case of the parity of the sum of an even and an odd number?

? Look at the following expressions and the visualisation. Write the corresponding explanation and examples.

Explanation with Algebra and Visualisation			Examples
	$4p$ and $(4q + 2)$ $= 4p + (4q + 2)$ $= 4p + 4q + 2$ $= 4(p + q) + 2$.		

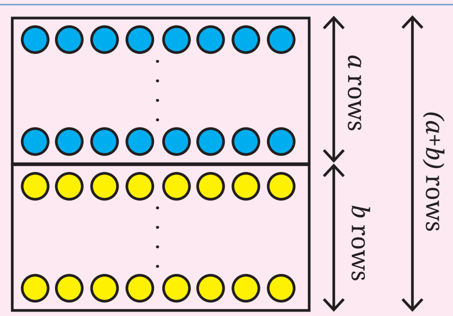
Notice how we are able to generalise and prove properties of arithmetic using algebra and also using visualisation.

Always, Sometimes, or Never

We examine different statements about factors and multiples and determine whether a statement is 'Always True', 'Sometimes True', or 'Never True'.

We know that the sum of any two multiples of 2 is also a multiple of 2.

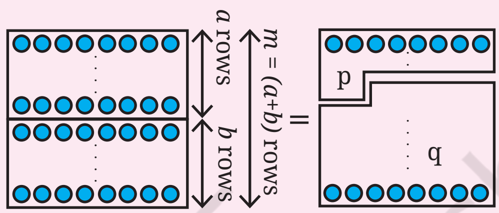
1. If 8 exactly divides two numbers separately, it must exactly divide their sum.

Explanation with Algebra and Visualisation			Examples
The two numbers have 8 as a factor; in other words, the two numbers are multiples of 8.	$8a$ and $8b$.		8 and 16. 16 and 56. 80 and 120.
As multiples of 8 are obtained by repeatedly adding 8, the sum of two multiples of 8 will also be a multiple of 8.	$8a + 8b$ $= 8(a + b)$.		$8 + 16 = 8(1 + 2)$ $= 24$. $16 + 56 = 72$. $80 + 120 = 200$.

Statement 1 is always true. Determine if it is true with subtraction.

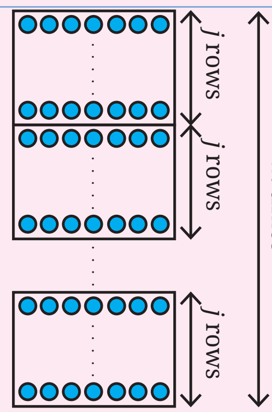
In general, **if a divides M and a divides N, then a divides $M + N$ and a divides $M - N$. In other words, if M and N are multiples of a, then $M + N$ and $M - N$ will also be multiples of a.**

2. If a number is divisible by 8, then 8 also divides any two numbers (separately) that add up to the number.

Explanation with Algebra and Visualisation			Examples
A number divisible by 8 is a multiple of 8.	$8m$		8, 16, 56, 72.
A number divisible by 8 can be expressed as a sum of two multiples of 8 or sum of two non-multiples of 8.	$8m = 8a + 8b$ $8m = p + q$ (p, q not multiples of 8)	 	72 = 48 + 24 (8×9 = 8×6 + 8×3). 72 = 50 + 22

So, statement 2 is sometimes true.

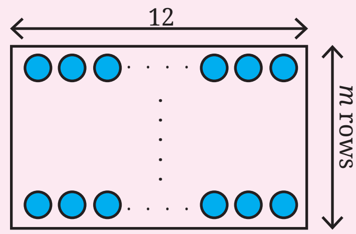
3. If a number is divisible by 7, then all multiples of that number will be divisible by 7.

Explanation with Algebra and Visualisation			Examples
Numbers divisible by 7 will have 7 as a factor.	$7j$		$14 = 7 \times 2$ ($j = 2$). $42 = 7 \times 6$ ($j = 6$). $98 = 7 \times 14$ ($j = 14$).
This contains a total of mj rows. So this is also a multiple of 7.	$(7j) \times m$	 	Some multiples of 14: $28 = (7 \times 2) \times 2$. $70 = (7 \times 2) \times 5$. $154 = (7 \times 2) \times 11$

The number $7jm$ or ($7 \times j \times m$) has a factor of 7. We can see that Statement 3 is always true.

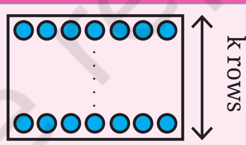
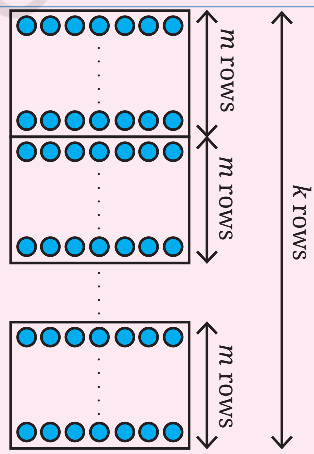
In general, **if *A* is divisible by *k*, then all multiples of *A* are divisible by *k*.**

4. If a number is divisible by 12, then the number is also divisible by all the factors of 12.

Explanation with Algebra and Visualisation			Examples
A number divisible by 12 is a multiple of 12.	$12m$		12, 24, 36, 48, 108, 132.
Factors of multiples of 12 will include factors of 12.	$12m$ $= 2 \times 6 \times m$ $= 3 \times 4 \times m$	 A factor of 12 covers a row fully. Hence, it covers all multiples of 12 fully.	Factors of 24: 1, 2, 3, 4, 6, 8, 12, 24.

In general, **if *A* is divisible by *k*, then *A* is divisible by all the factors of *k*.** Hence, Statement 4 is always true.

5. If a number is divisible by 7, then it is also divisible by any multiple of 7.

Explanation with Algebra and Visualisation			Examples
Numbers divisible by 7 are multiples of 7.	$7k$		
Multiples of 7. $7k$ will be divisible by $7m$ if and only if m is a factor of k.	$7m$ *If* $k = ym$ then $7k \div 7m = 7ym \div 7m = y$		42 (7 × 6) is divisible by 7 but it is not divisible by 28 (7 × 4). 42 (7 × 6) is divisible by 7 and it is divisible by 14 (7 × 2).

We can see that this statement is only sometimes true.

Math Talk

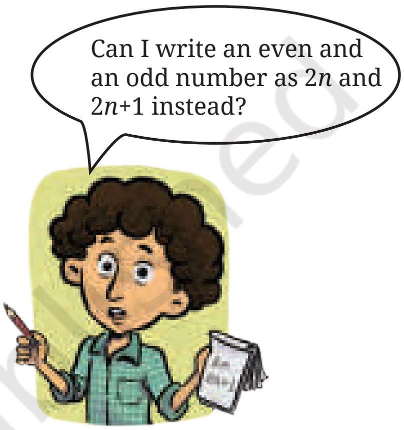
? Examine each of the following statements, and determine whether it is 'Always true', 'Sometimes true', 'Never true'.

? 6. If a number is divisible by both 9 and 4, it must be divisible by 36.

? 7. If a number is divisible by both 6 and 4, it must be divisible by 24.

In general, if A is divisible by k and A is also divisible by m, then A is divisible by the LCM of k and m. This is because A is a multiple of k and also a multiple of m, so A's prime factorisation should contain the prime factorisation of LCM (k, m).

? 8. When you add an odd number to an even number we get a multiple of 6.



We know that multiples of 6 are all even numbers. The sum of an odd number and an even number will be an odd number. Therefore, this statement is never true. We can also explain this algebraically. Suppose,

$$(2n) + (2m + 1) = 6j,$$

where $2n$ is an even number, $2m + 1$ is an odd number, and $6j$ is a multiple of 6. Then

$$2n + 2m = 6j - 1$$

$$2\,(n + m) = 6j - 1$$

which means $2(n + m)$, which is an even number, should be equal to $6j - 1$, which is an odd number. This is never true.

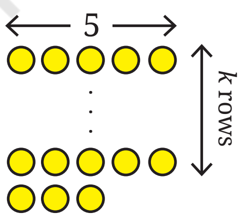
What Remains?

? Find a number that has a remainder of 3 when divided by 5. Write more such numbers.

? Which algebraic expression(s) capture all such numbers?

(i) $3k + 5$ (ii) $3k - 5$ (iii) $\frac{3k}{5}$ (iv) $5k + 3$ (v) $5k - 2$ (vi) $5k - 3$

The numbers that leave a remainder of 0 when divided by 5 are the multiples of 5. But we want numbers that leave a remainder of 3 when divided by 5. These numbers are 3 more than multiples of 5. Multiples of 5 are of the form $5k$. So, numbers that leave a remainder of 3 when divided by 5 are those of the form $5k + 3$



k =	0	1	2	3	4
$5k + 3$ =	3	8	13	18	23

Let us consider another expression, $5k - 2$, and see the values it takes for different values of k.

Numbers that leave a remainder of 3 when divided by 5 can also be seen as 2 less than multiples of 5; $5k - 2$, where $k \geq 1$.

$k =$	1	2	3	4	5
$5k - 2 =$	3	8	13	18	23

Are there other expressions that generate numbers that are 3 more than a multiple of 5?

Figure it Out

1. The sum of four consecutive numbers is 34. What are these numbers?
2. Suppose p is the greatest of five consecutive numbers. Describe the other four numbers in terms of p.
3. For each statement below, determine whether it is always true, sometimes true, or never true. Explain your answer. Mention examples and non-examples as appropriate. Justify your claim using algebra.
   (i) The sum of two even numbers is a multiple of 3.
   (ii) If a number is not divisible by 18, then it is also not divisible by 9.
   (iii) If two numbers are not divisible by 6, then their sum is not divisible by 6.
   (iv) The sum of a multiple of 6 and a multiple of 9 is a multiple of 3.
   (v) The sum of a multiple of 6 and a multiple of 3 is a multiple of 9.
4. Find a few numbers that leave a remainder of 2 when divided by 3 and a remainder of 2 when divided by 4. Write an algebraic expression to describe all such numbers.
5. "I hold some pebbles, not too many,
   When I group them in 3's, one stays with me.
   Try pairing them up — it simply won't do,
   A stubborn odd pebble remains in my view.
   Group them by 5, yet one's still around,
   But grouping by seven, perfection is found.
   More than one hundred would be far too bold,
   Can you tell me the number of pebbles I hold?"

6. Tathagat has written several numbers that leave a remainder of 2 when divided by 6. He claims, "If you add any three such numbers, the sum will always be a multiple of 6." Is Tathagat's claim true?

7. When divided by 7, the number 661 leaves a remainder of 3, and 4779 leaves a remainder of 5. Without calculating, can you say what remainders the following expressions will leave when divided by 7? Show the solution both algebraically and visually.

   (i) 4779 + 661 (ii) 4779 – 661

8. Find a number that leaves a remainder of 2 when divided by 3, a remainder of 3 when divided by 4, and a remainder of 4 when divided by 5. What is the smallest such number? Can you give a simple explanation of why it is the smallest?

5.2 Checking Divisibility Quickly

Earlier, you have learnt shortcuts to check whether a given number, written in the Indian number system is divisible by 2, 4, 5, 8, and 10. Let us revisit them.

Divisibility by 10, 5, and 2: If the units digit of a number is '0', then it is divisible by 10. Let us understand why this works through algebra.

We can write the general form of a number in the Indian system using a set of letter-numbers. For example, a 5-digit number can be expressed as, *edcba* denoting $e \times 10000 + d \times 1000 + c \times 100 + b \times 10 + a$. The letter-numbers e, d, c, b, and a denote each digit of a 5-digit number.

Any number can be written in general as ... *dcba*, where the letter-numbers a, b, c and d represent the units, tens, hundreds and thousands digit, respectively, and so on. As a sum of place values, this number is —

$$\ldots + 1000d + 100c + 10b + a.$$

(For example, in the number 4075, $d = 4$, $c = 0$, $b = 7$, and $a = 5$.)

We know that each place value, with the exception of the units place, is a multiple of 10. So, $10b$, $100c$, ... all will be multiples of 10. Hence, the number will be divisible by 10 if and only if the units digit a is 0.

? Similarly, explain using algebra why the divisibility shortcuts for 5, 2, 4, and 8 work.

Let us now examine shortcuts to check divisibility by some other numbers and explain why they work!

A Shortcut for Divisibility by 9

? Can you say, without actually calculating, which of these numbers are divisible by 9: 999, 909, 900, 90, 990?

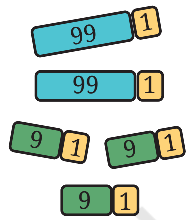
All of them.

? Can we say that any number made up of only the digits '0' and '9', in any order, will always be divisible by 9?

Yes, if each digit is either 0 or 9, then each term in its expanded form will be 9 × □ or 0 × □ (the '□' denotes a place value). This means each term will be a multiple of 9, for example,

$$99009 = 9 \times 10000 + 9 \times 1000 + 0 \times 100 + 0 \times 10 + 9 \times 1.$$

But this shortcut alone cannot identify all the multiples of 9. Unlike the numbers 2, 5, and 10, we cannot identify the multiples of 9 by just looking at the unit's digit. 99 and 109 are two numbers with 9 as the units digit; but 99 is divisible by 9, while 109 is not.



? Is 10 divisible by 9? If not, what is the remainder?

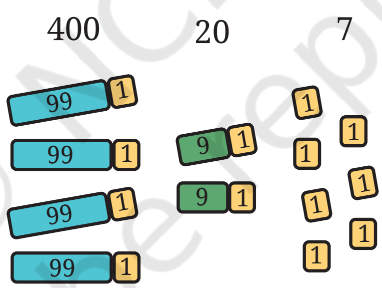
Check the divisibility of other multiples of 10 (10, 20, 30, ...) by 9.

You will notice that for any multiple of 10, the remainder is the same as the number of tens.

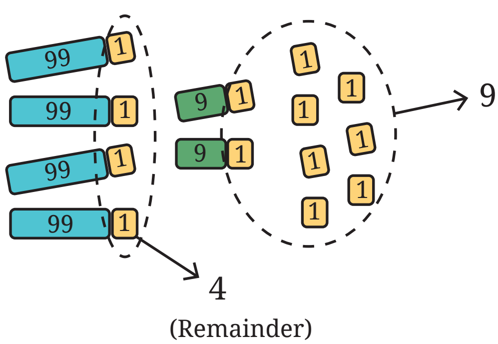
? Similarly, look at the remainder when the multiples of 100 (100, 200, 300, ...) are divided by 9. What do you notice?

The remainder is the same as the number of hundreds for any multiple of 100.

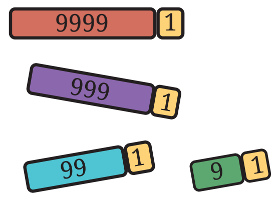
? Using this observation, find the remainder when 427 is divided by 9.



We see that 427 has 4 hundreds; thus, its corresponding remainder (upon division by 9) would be 4. 427 has 2 tens, and its corresponding remainder would be 2. We have 7 units also remaining. Adding all the remainders, we get 4 + 2 + 7 = 13. We can make one more group of 9 with 13, leaving a remainder of 4. Therefore, 427 ÷ 9 gives a remainder of 4.



? Will this work with bigger numbers?



You can see that this is true for any place value:

1 = 0 + 1

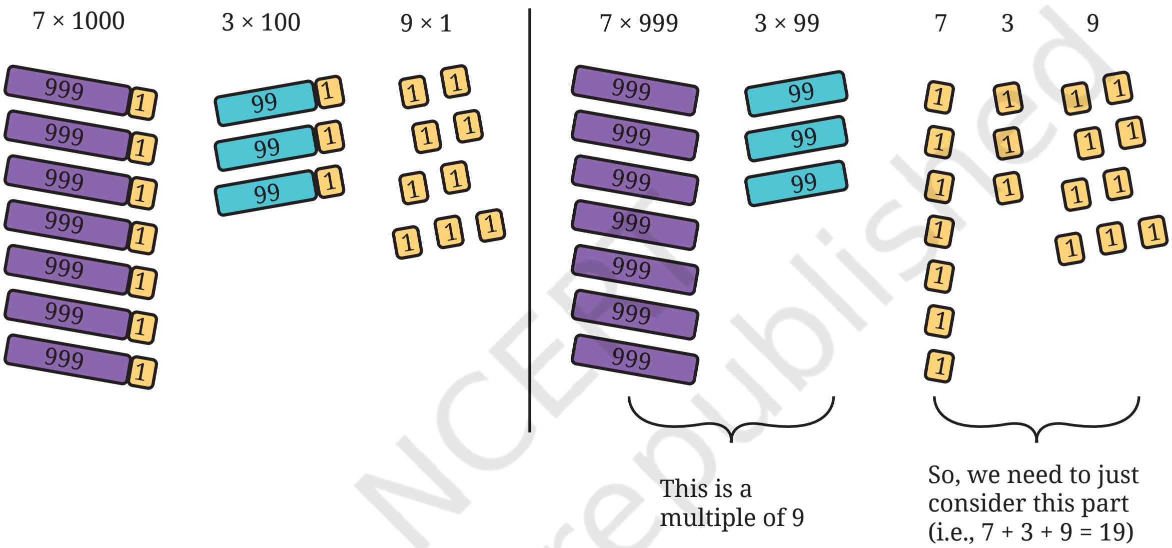
10 = 9 + 1

100 = 99 + 1

1000 = 999 + 1

10000 = 9999 + 1, and so on. Each digit thus denotes the remainder when the corresponding place value is divided by 9.

For example, to find the remainder of 7309 when divided by 9, we can just add all the digits—7 + 3 + 0 + 9—to get 19. This can be seen as follows:



$7 \times 1000 + 3 \times 100 + 0 \times 10 + 9 \times 1$

$= 7 \times (999 + 1) + 3 \times (99 + 1) + 0 \times (9 + 1) + 9 \times (0+1)$

$= (7 \times 999 + 3 \times 99 + 0 \times 9 + 9 \times 0) + (7 \times 1 + 3 \times 1 + 0 \times 1 + 9 \times 1)$

$= \underline{(7 \times 999 + 3 \times 99 + 0 \times 9 + 9 \times 0)} + \underline{(7 + 3 + 0 + 9)}$.

This is a multiple of 9 — So, we need to just consider this part

This means that the number 7309 is 19 more than some multiple of 9. The digits 1 and 9 can further be added to get 1 + 9 = 10. Now, we can say that 7309 is 10 more than a multiple of 9. And repeating this step for the number 10, we get the remainder to be 1 + 0 = 1, meaning 7309 is 1 more than a multiple of 9. Therefore, 7309 ÷ 9 gives a remainder of 1.

A number is divisible by 9 if and only if the sum of its digits is divisible by 9. Also, we can add the digits of a number repeatedly till a single digit is obtained. This single digit is the remainder when the number is divided by 9.

? Look at each of the following statements. Which are correct and why?

(i) If a number is divisible by 9, then the sum of its digits is divisible by 9.

(ii) If the sum of the digits of a number is divisible by 9, then the number is divisible by 9.

(iii) If a number is not divisible by 9, then the sum of its digits is not divisible by 9.

(iv) If the sum of the digits of a number is not divisible by 9, then the number is not divisible by 9.

Learning maths is not just about knowing some shortcuts and following procedures but about understanding 'why' something works.

Figure it Out

1. Find, without dividing, whether the following numbers are divisible by 9.

   (i) 123 (ii) 405 (iii) 8888 (iv) 93547 (v) 358095

2. Find the smallest multiple of 9 with no odd digits.

3. Find the multiple of 9 that is closest to the number 6000.

4. How many multiples of 9 are there between the numbers 4300 and 4400?

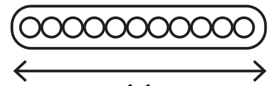
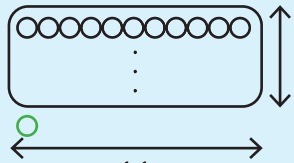
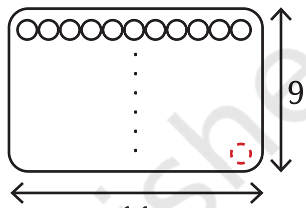
A Shortcut for Divisibility by 3

We know that all the multiples of 9 are also multiples of 3. That is, if a number is divisible by 9, it will also be divisible by 3. However, there are other multiples of 3 that are not multiples of 9 for example— 15, 33, and 87.

The shortcut to find the divisibility by 3 is similar to the method for 9. A number is divisible by 3 if the sum of its digits is divisible by 3. Explore the remainders when powers of 10 are divided by 3. Explain why this method works.

A Shortcut for Divisibility by 11

Interestingly, the shortcut for 11 is also based on checking the remainders with place value. Let us see how.

Units place (1)	$11 \times 0 = 0$ $1 = 11 \times 0 + \mathbf{1}$	1 is one more than a multiple of 11.	
Tens place (10)	$11 \times 1 = 11$ $10 = 11 \times 1 - \mathbf{1}$	10 is one less than a multiple of 11.	
Hundreds place (100)	$11 \times 9 = 99$ $100 = 11 \times 9 + \mathbf{1}$	100 is one more than a multiple of 11.	
Thousands place (1000)	$11 \times 91 = 1001$ $1000 = 11 \times 91 - \mathbf{1}$	1000 is one less than a multiple of 11.	
⋮	⋮	⋮	⋮

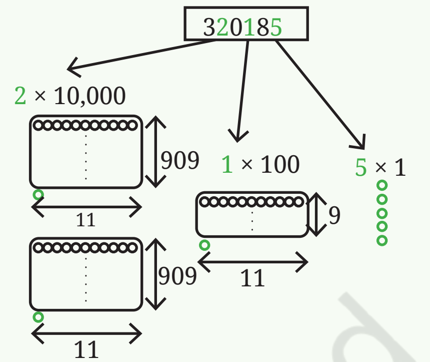
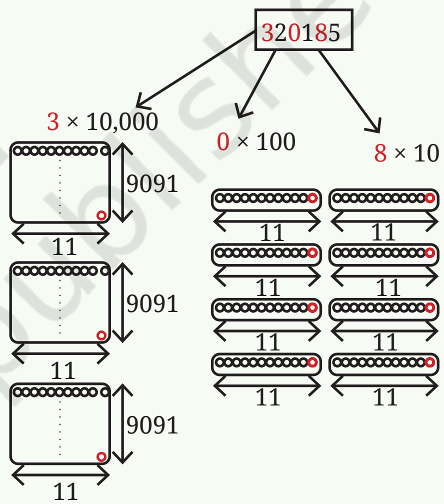
This alternating pattern of one more than 11 and one less than 11 continues for higher place values.

Since 400 contains 4 hundreds, 400 is 4 more than a multiple of 11 (396 + 4). Since 60 contains 6 tens, 60 is 6 less than a multiple of 11 (66 – 6). Since 2 contains 2 units, 2 is 2 more than a multiple of 11, i.e., 2 = (0 + 2).

? Using these observations, can you tell whether the number 462 is divisible by 11?

? What could be a general method or shortcut to check divisibility by 11?

We saw that the place values alternate as 1 more and 1 less than a multiple of 11. Using this observation,

Steps	Purpose	Example for the Number 320185
1. Add the digits of place values which are 1 more (than a multiple of 11), i.e., place values corresponding to 1, 100, 10000, and so on.	To know how much in excess we are with respect to a multiple of 11 for these place values.	 Total excess, 2 + 1 + 5 = 8.
2. Add the digits of place values which are 1 less (than a multiple of 11), i.e., place values corresponding to 10, 1000, 100000, and so on.	To know how short we are with respect to a multiple of 11 for these place values.	 Total short, 3 + 0 + 8 = 11.
3. Compute the difference between these two sums, i.e., (number in excess) – (number short).	To know the remainder obtained when divided by 11.	 8 – 11 = – 3. (3 short of a multiple of 11)

The difference between these two sums 8 – 11 = – 3, indicating that the number 3,28,105 is 3 short of or 8 more than a multiple of 11.

? If this difference is 11 or a multiple of 11, what does that say about the remainder obtained when the number is divisible by 11?

? Using this shortcut, find out whether the following numbers are divisible by 11. Further, find the remainder if the number is not divisible by 11.

(i) 158 (ii) 841 (iii) 481 (iv) 5529 (v) 90904 (vi) 857076

Look at the following procedure—

Steps to follow	Example for the number 328105
1. Place alternating '+' and '–' signs before every digit starting from the unit's digit.	$-3 + 2 - 8 + 1 - 0 + 5$
2. Evaluate the expression.	$-3 + 2 - 8 + 1 - 0 + 5 = -3$
3. The result denotes the remainder obtained when the number is divided by 11.	328105 is 3 less than or 8 more than a multiple of 11

? Is this method similar to or different from the method we saw just before?

Math Talk

? Fill in the following table. Find a quick way to do this?

Number	Divisible by								
	2	3	4	5	6	8	9	10	11
128	Yes	No	No	No	No	Yes	No	No	No
990									
1586									
275									
6686									
639210									
429714									
2856									
3060									
406839									

More on Divisibility Shortcuts

Divisibility Shortcuts for Other Numbers

? How can we find out if a number is divisible by 6?

? Will checking its divisibility by its factors 2 and 3 work? Use the shortcuts for 2 and 3 on these numbers and divide each number by 6 to verify— 38, 225, 186, 64.

? How about checking divisibility by 24? Will checking the divisibility by its factors, 4 and 6, work? Why or why not?

Determining divisibility by 24 by checking divisibility by 4 and by 6 does not work. For example, the number 12 is divisible by both 4 and 6, but not by 24.

To check for the divisibility by 24, we can instead check for the divisibility by 3 and divisibility by 8.

Explain using prime factorisation why checking divisibility by 3 and 8 works for checking divisibility by 24, but checking divisibility by 4 and 6 is not sufficient for checking divisibility by 24.

There are such shortcuts to check divisibility by every number until 100, and for some numbers beyond 100. You may try to understand how these work after learning certain concepts in higher grades.

Digital Roots

Take a number. Add its digits repeatedly till you get a single-digit number. This single-digit number is called the digital root of the number. For example, the digital root of the number 489710 will be

$$2\ (4 + 8 + 9 + 7 + 1 + 0 = 29,\ 2 + 9 = 11,\ 1 + 1 = 2).$$

? What property do you think this digital root will have? Recall that we did this while finding the divisibility shortcut for 9.

? Between the numbers 600 and 700, which numbers have the digital root: (i) 5, (ii) 7, (iii) 3?

? Write the digital roots of any 12 consecutive numbers. What do you observe?

Math Talk

We saw that the digital root of multiples of 9 is always 9.

? Now, find the digital roots of some consecutive multiples of (i) 3, (ii) 4, and (iii) 6.

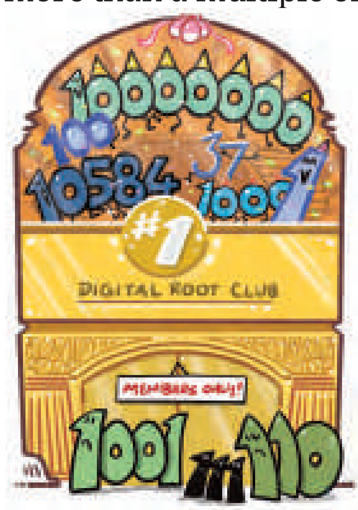
? What are the digital roots of numbers that are 1 more than a multiple of 6? What do you notice?

Try to explain the patterns noticed.

? I'm made of digits, each tiniest and odd,
No shared ground with root #1—how odd!

My digits count, their sum, my root—
All point to one bold number's pursuit—
The largest odd single-digit I proudly claim.

What's my number? What's my name?



Aryabhata II's (c. 950 CE) work *Mahāsiddhānta*, mentions the method of computing the digital root of a number by repeatedly adding the digits till a single-digit number is obtained. This method is known to have been used to perform checks on calculations of arithmetic operations.

Figure it Out

1. The digital root of an 8-digit number is 5. What will be the digital root of 10 more than that number?

2. Write any number. Generate a sequence of numbers by repeatedly adding 11. What would be the digital roots of this sequence of numbers? Share your observations.

3. What will be the digital root of the number $9a + 36b + 13$?

4. Make conjectures by examining if there are any patterns or relations between
   (i) the parity of a number and its digital root.
   (ii) the digital root of a number and the remainder obtained when the number is divided by 3 or 9.

Math Talk

5.3 Digits in Disguise

Last year, we saw cryptarithms — puzzles where each letter stands for a digit, each digit is represented by at most one letter, and the first digit of a number is never 0.

Solve the cryptarithms given below.

(i) $\begin{array}{r} \text{A1} \\ +\ \text{1B} \\ \hline \text{B0} \end{array}$ (ii) $\begin{array}{r} \text{AB} \\ +\ \text{37} \\ \hline \text{6A} \end{array}$ (iii) $\begin{array}{r} \text{ON} \\ \text{ON} \\ +\ \text{ON} \\ \hline \text{PO} \end{array}$ (iv) $\begin{array}{r} \text{QR} \\ \text{QR} \\ +\ \text{QR} \\ \hline \text{PRR} \end{array}$

Let us now try solving some cryptarithms involving multiplication.

(v) PQ × 8 = RS.

Guna says, "Oh, this means a 2-digit number multiplied by 8 should give another 2-digit number. I know that 10 × 8 = 80. But the units digits of 10 and 80 are the same, which we don't want. For the same reason PQ cannot be 11 as P and Q correspond to different digits. 12 × 8 = 96 fits all the conditions". Can PQ be 13? Think.

It is not possible because 13 × 8 = 104. For all 2-digit numbers greater than 12, the product with 8 is a 3-digit number.

(vi) Try this now: GH × H = 9K.

This means a 2-digit number multiplied by a 1-digit number gives another 2-digit number in the 90s. Observe the letters corresponding to the units digits in this cryptarithm. Pick the solution to this question from the options given below:

11 × 9 = 99, 12 × 8 = 96, 46 × 2 = 92, 24 × 4 = 96, 47 × 2 = 94, 31 × 3 = 93, 16 × 6 = 96.

(vii) Here is one more: BYE × 6 = RAY.

Anshu says, "Since the product is a 3-digit number, B can't be 2 or more. If B = 2, i.e., 2 hundreds, the product will be more than 1200. So, B = 1."

What can you say about 'Y'? What digits are possible/not possible?

"Y cannot be 7 or more because, if Y = 7, then 170 × 6 = 1020; but we want a 3-digit product. Also, Y will be even", Anshu explains.

We can solve cryptarithms using patterns, properties, and reasoning related to numbers and operations.

Solve the following:

(i) UT × 3 = PUT	(ii) AB × 5 = BC	(iii) L2N × 2 = 2NP
(iv) XY × 4 = ZX	(v) PP × QQ = PRP	(vi) JK × 6 = KKK

Figure it Out

1. If 31*z*5 is a multiple of 9, where *z* is a digit, what is the value of *z*? Explain why there are two answers to this problem.

2. "I take a number that leaves a remainder of 8 when divided by 12. I take another number which is 4 short of a multiple of 12. Their sum will always be a multiple of 8", claims Snehal. Examine his claim and justify your conclusion.

3. When is the sum of two multiples of 3, a multiple of 6 and when is it not? Explain the different possible cases, and generalise the pattern.

4. Sreelatha says, "I have a number that is divisible by 9. If I reverse its digits, it will still be divisible by 9".
   (i) Examine if her conjecture is true for any multiple of 9.
   (ii) Are any other digit shuffles possible such that the number formed is still a multiple of 9?

5. If 48*a*23*b* is a multiple of 18, list all possible pairs of values for *a* and *b*.

6. If $3p7q8$ is divisible by 44, list all possible pairs of values for p and q.

7. Find three consecutive numbers such that the first number is a multiple of 2, the second number is a multiple of 3, and the third number is a multiple of 4.
   Are there more such numbers? How often do they occur?

   Try This

8. Write five multiples of 36 between 45,000 and 47,000. Share your approach with the class.

   Math Talk

9. The middle number in the sequence of 5 consecutive even numbers is $5p$. Express the other four numbers in sequence in terms of p.

10. Write a 6-digit number that it is divisible by 15, such that when the digits are reversed, it is divisible by 6.

11. Deepak claims, "There are some multiples of 11 which, when doubled, are still multiples of 11. But other multiples of 11 don't remain multiples of 11 when doubled". Examine if his conjecture is true; explain your conclusion.

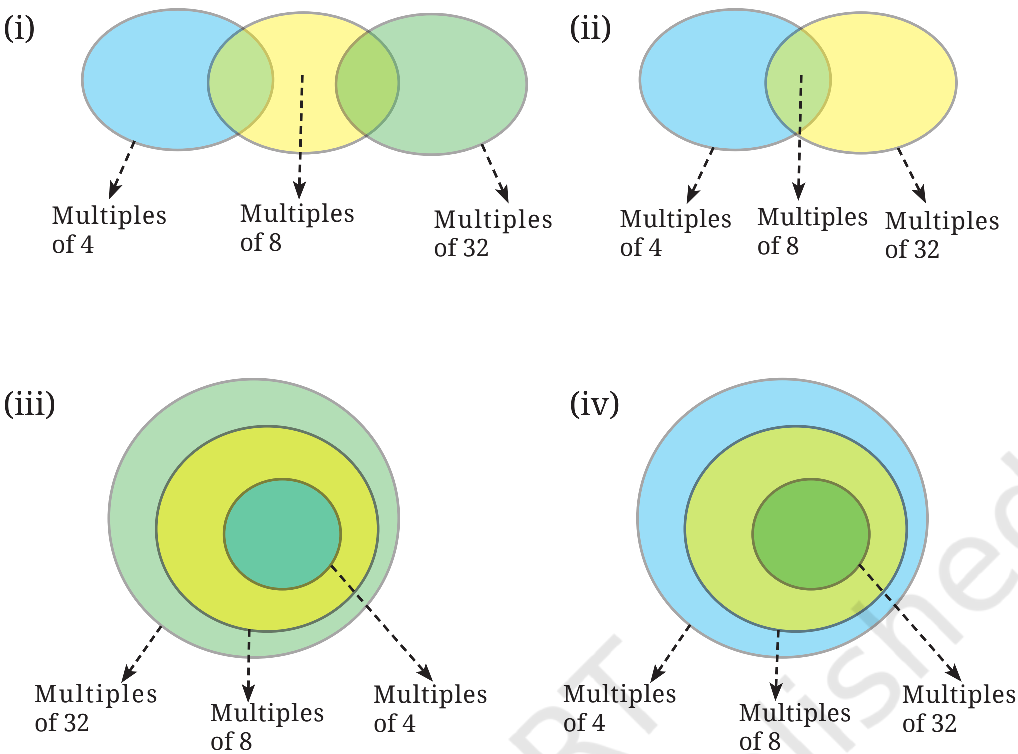
12. Determine whether the statements below are 'Always True', 'Sometimes True', or 'Never True'. Explain your reasoning.
    (i) The product of a multiple of 6 and a multiple of 3 is a multiple of 9.
    (ii) The sum of three consecutive even numbers will be divisible by 6.
    (iii) If *abcdef* is a multiple of 6, then *badcef* will be a multiple of 6.
    (iv) $8(7b - 3) - 4(11b + 1)$ is a multiple of 12.

13. Choose any 3 numbers. When is their sum divisible by 3? Explore all possible cases and generalise.

14. Is the product of two consecutive integers always multiple of 2? Why? What about the product of these consecutive integers? Is it always a multiple of 6? Why or why not? What can you say about the product of 4 consecutive integers? What about the product of five consecutive integers?

15. Solve the cryptarithms —
    (i) EF × E = GGG (ii) WOW × 5 = MEOW

16. Which of the following Venn diagrams captures the relationship between the multiples of 4, 8, and 32?



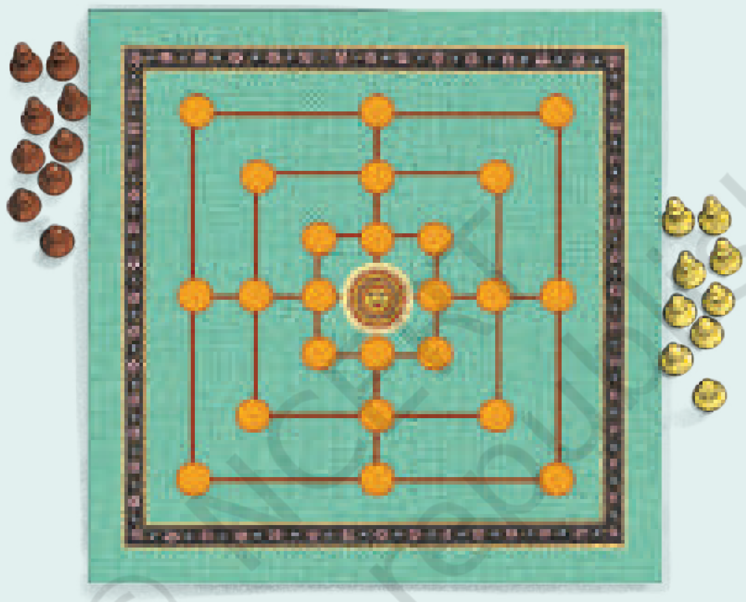
SUMMARY

- We explored and learnt various properties of divisibility—
  - If a is divisible by b, then all multiples of a are divisible by b.
  - If a is divisible by b, then a is divisible by all the factors of b.
  - If a divides m and a divides n, then a divides $m + n$ and $m - n$.
  - If a is divisible by b and is also divisible by c, then a is divisible by the LCM of b and c.
- We learnt shortcuts to check divisibility by 3, 9 and 11, and why they work.
- Through all this we were exposed to the power of mathematical thinking and reasoning, using algebra, visualisation, examples and counterexamples.

It's Puzzle Time!

Navakankari

Navakankari, also known as *Sālu Mane Āṭa*, *Chār-Pār*, or *Navkakri*, is a traditional Indian board game that is the same as 'Nine Men's Morris' or 'Mills in the West'. It is a strategy game for two players where the goal is to form lines of three pawns to eliminate the opponent's pawns or block their movement.



Gameplay

1. Each player starts with 9 pawns. The players take turns in placing their pawns on the marked intersections. An intersection can have at most one pawn.
2. Once all the pawns are placed, the players take turns to move one of their pawns to adjacent empty intersections to form lines of three. The line can be horizontal or vertical.
3. Once a player makes a line with their pawns they can remove any one of the opponent's pawns as long as it is not a part of one of their lines.

A player wins if the opponent has less than 3 pawns or is unable to make a move.

6 WE DISTRIBUTE, YET THINGS MULTIPLY

0874CH06

We have seen how algebra makes use of letter symbols to write general statements about patterns and relations in a compact manner. Algebra can also be used to justify or prove claims and conjectures (like the many properties you saw in the previous chapter) and to solve problems of various kinds.

Distributivity is a property relating multiplication and addition that is captured concisely using algebra. In this chapter, we explore different types of multiplication patterns and show how they can be described in the language of algebra by making use of distributivity.

6.1 Some Properties of Multiplication

Increments in Products

Consider the multiplication of two numbers, say, 23×27.

? 1. By how much does the product increase if the first number (23) is increased by 1?

? 2. What if the second number (27) is increased by 1?

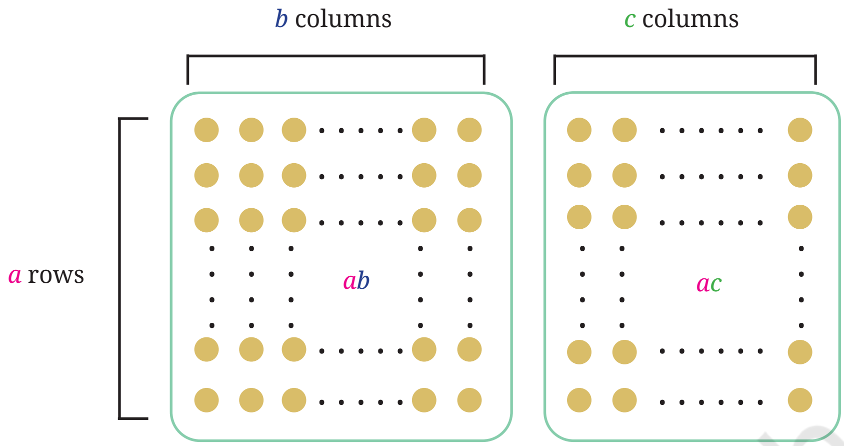
? 3. How about when both numbers are increased by 1?

? Do you see a pattern that could help generalise our observations to the product of any two numbers?

Let us first consider a simpler problem—find the increase in the product when 27 is increased by 1. From the definition of multiplication (and the commutative property), it is clear that the product increases by 23. This can be seen from the distributive property of multiplication as well. If a, b and c are three numbers, then—

$$a\,(b + c) = ab + ac$$

This property can be visualised nicely using a diagram:



$$a\,(b + c)$$

This is called the **distributive property** of multiplication over addition. Using the identity $a\,(b + c) = ab + ac$ with $a = 23$, $b = 27$, and $c = 1$, we have

$$23\,(27 + 1) = 23 \times 27 + \boxed{23}$$

Increase

Remember that here, $a\,(b + c)$ and 23 (27 + 1) mean $a \times (b + c)$, and $23 \times (27 + 1)$, respectively. We usually skip writing the '×' symbol before or after brackets, just as in the case of expressions like $5a$, xy, etc.

We can also similarly expand $(a + b)\,c$ using the distributive property as follows—

$(a + b)\,c = c\,(a + b)$ (commutativity of multiplication)

$= ca + cb$ (distributivity)

$= ac + bc$ (commutativity of multiplication)

We can use the distributive property to find, in general, how much a product increases if one or both the numbers in the product are increased by 1. Suppose the initial two numbers are a and b. If one of the numbers, say b, is increased by 1, then we have—

$$a\,(b + 1) = ab \times \boxed{a}$$

Increase

Now let us see what happens if both numbers in a product are increased by 1. If in a product ab, both a and b are increased by 1, then we obtain $(a + 1)\,(b + 1)$.

? How do we expand this?

Let us consider $(a + 1)$ as a single term. Then, by the distributive property, we have

$(a + 1)(b + 1) = (a + 1)\,b + (a + 1)\,1$

Again applying the distributive property, we obtain

$$(a + 1)(b + 1) = (a + 1)\,b + (a + 1)\,1$$
$$= ab + \boxed{(b + a + 1)}$$

Increase

If $a = 23$, and, $b = 27$, we get

$$(23 + 1)(27 + 1) = (23 + 1)\,27 + (23 + 1)\,1$$
$$= 23 \times 27 + \boxed{(27 + 23 + 1)}$$

Increase

Thus, the product ab increases by $a + b + 1$ when each of a and b are increased by 1.

? What would we get if we had expanded $(a + 1)(b + 1)$ by first taking $(b + 1)$ as a single term? Try it?

? What happens when one of the numbers in a product is increased by 1 and the other is decreased by 1? Will there be any change in the product?

Let us again take the product ab of two numbers a and b. If a is increased by 1 and b is decreased by 1, then their product will be $(a + 1)(b – 1)$. Expanding this, we get

$$(a + 1)(b - 1) = (a + 1)\,b - (a + 1)\,1$$
$$= ab + b - (a + 1)$$
$$= ab + \boxed{b - a - 1}$$

Increase

If $a = 23$, and $b = 27$, we get

$$(23 + 1)(27 - 1) = (23 + 1)\,27 - (23 + 1)\,1$$
$$= 23 \times 27 + 27 - (23 + 1)$$
$$= 23 \times 27 + \boxed{27 - 23 - 1}$$

Increase

? Will the product always increase? Find 3 examples where the product decreases.

? What happens when a and b are negative integers?

Check by substituting different values for a and b in each of the above cases. For example, $a = –5$, $b = 8$; $a = –4$, $b = –5$; etc.

We have seen that integers also satisfy the distributive property, that is, if x, y and z are any three integers, then $x(y + z) = xy + xz$.

Thus, the expressions we have for increase of products hold when the letter-numbers take on negative integer values as well.

Recall that two algebraic expressions are equal if they take on the same values when their letter-numbers are replaced by numbers. These numbers could be any integers. Mathematical statements that express the equality of two algebraic expressions, such as

$$a(b + 8) = ab + 8a,$$

$$(a + 1)(b - 1) = ab + b - a - 1, \text{ etc.,}$$

are called **identities.**

? By how much will the product of two numbers change if one of the numbers is increased by m and the other by n?

If a and b are the initial numbers being multiplied, they become $a + m$ and $b + n$.

$$(a + m)(b + n) = (a + m)b + (a + m)n$$

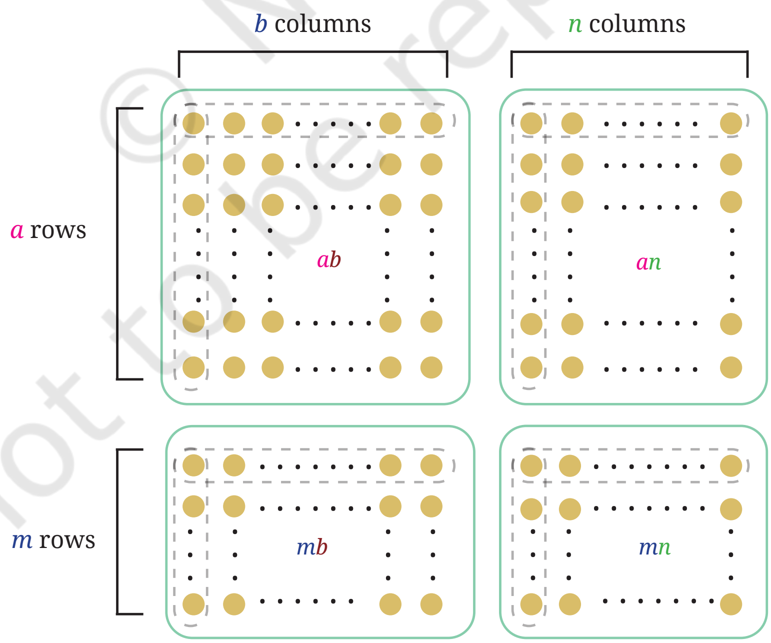
$$= ab + mb + an + mn$$

The increase is $an + bm + mn$.

Notice that the product is the sum of the product of each term of $(a + m)$ with each term of $(b + n)$.

Identity 1 $(a + m)(b + n) = ab + mb + an + mn$

This identity can be visualised as follows—



$(a + m)(b + n)$

? This identity can be used to find how products change when the numbers being multiplied are increased or decreased by any amount. Can you see how this identity can be used when one or both numbers are decreased?

For example, let us reconsider the case when one number is increased by 1 and the other decreased by 1. Let us write the product $(a + 1)(b - 1)$ as $(a + 1)(b + (-1))$. Taking $m = 1$ and $n = -1$ in Identity 1, we have

$$ab + (1) \times b + a \times (-1) + (1) \times (-1) = ab + b - a - 1,$$

which is the same expression that we obtain earlier.

? Use Identity 1 to find how the product changes when

(i) one number is decreased by 2 and the other increased by 3;

(ii) both numbers are decreased, one by 3 and the other by 4.

? Verify the answers by finding the products without converting the subtractions to additions.

Generalising this, we can find the product $(a + u)(b - v)$ as follows.

$$(a + u)(b - v) = (a + u)\,b - (a + u)\,v$$

$$= ab + ub - (av + uv)$$

$$= ab + ub - av - uv.$$

Check that this is the same as taking $m = u$ and $n = -v$ in Identity 1.

As in Identity 1, the product $(a + u)(b - v)$ is the sum of the product of each term of $a + u$ (a and u) with each term of $b - v$ (b and $(-v)$). Notice that the signs of the terms in the products can be determined using the usual rules of integer multiplication.

See how the rules of integer multiplication allows us to handle multiple cases using a single identity!

? Expand (i) $(a - u)(b + v)$, (ii) $(a - u)(b - v)$.

We get

$$(a - u)(b + v) = ab - ub + av - uv, \text{ and}$$

$$(a - u)(b - v) = ab - ub - av + uv.$$

The distributive property is not restricted to two terms within a bracket.

? **Example 1:** Expand $\frac{3a}{2}(a - b + \frac{1}{5})$.

$$\frac{3a}{2}(a - b + \frac{1}{5}) = (\frac{3a}{2} \times a) - (\frac{3a}{2} \times b) + (\frac{3a}{2} \times \frac{1}{5}).$$

The terms can be simplified as follows—

$$\frac{3a}{2} \times a = \frac{3}{2} \times (a \times a).$$

Using exponent notation, we can write $\frac{3}{2} \times (a \times a) = \frac{3}{2}a^2$.

$$\frac{3a}{2} \times b = \frac{3}{2} \times (a \times b) = \frac{3}{2}ab.$$

$$\frac{3a}{2} \times \frac{1}{5} = (\frac{3}{2} \times \frac{1}{5})\, a = \frac{3}{10}a$$

So we get

$$\frac{3a}{2}(a - \text{b} + \frac{1}{5}) = \frac{3}{2}a^2 - \frac{3}{2}ab + \frac{3}{10}a.$$

? Can any two terms be added to get a single term?

For example, can $\frac{3}{2}a^2$ and $\frac{3}{10}a$ be added to get a single term?

We see that no two terms have exactly the same letter-numbers, which would have allowed them to be simplified into a single term. So, a further simplification of the expression is not possible.
Recall that we call terms having the same letter-numbers **like terms.**

? **Example 2:** Expand $(a + b)(a + b)$.

We have $(a + b)(a + b) = (a+b)\, a + (a + b)b = a \times a + b \times a + ab + b \times b$

$$= a^2 + ba + ab + b^2$$

Since $ba = ab$, we have two terms having the same letter-numbers ab (or, that are like terms), and so can be added —

$$ba + ab = ab + ab = 2ab$$

So we get

$$(a + b)(a + b) = a^2 + 2ab + b^2.$$

? **Example 3:** Expand $(a + b)(a^2 + 2ab + b^2)$.

$(a + b)(a^2 + 2ab + b^2) = (a + b)a^2 + (a + b) \times 2ab + (a + b)b^2$

$= (a \times a^2) + ba^2 + (a \times 2ab) + (b \times 2ab) + ab^2 + (b \times b^2)$

The terms can be simplified as follows—

$$a \times a^2 = \text{a}^3 \text{ (why?)}$$

$$ba^2 = a^2b$$

$$a \times 2ab = 2 \times a \times a \times \text{b} = 2a^2b$$

$$b \times 2ab = 2 \times a \times b \times b = 2ab^2$$

$$b \times b^2 = b^3$$

$$\text{So, } (a + b)(a^2 + 2ab + b^2) = a^3 + a^2b + 2a^2b + 2ab^2 + ab^2 + b^3.$$

We see that a^2b and $2a^2b$ have the same letter-numbers (or, are like terms) and so can be added—

$$a^2b + 2a^2b = (1 + 2)\,a^2b = 3a^2b.$$

Similarly, ab^2 and $2ab^2$ are like terms and so can be added—

$$ab^2 + 2ab^2 = (1 + 2)ab^2 = 3ab^2.$$

Thus, we have

$$(a + b) \times (a^2 + 2ab + b^2) = a^3 + 3a^2b + 3ab^2 + b^3.$$

A Pinch of History

The distributive property of multiplication over addition was implicit in the calculations of mathematicians in many ancient civilisations, particularly in ancient Egypt, Mesopotamia, Greece, China, and India. For example, the mathematicians Euclid (in geometric form) and Āryabhaṭa (in algebraic form) used the distributive law in an implicit manner extensively in their mathematical and scientific works. The first explicit statement of the distributive property was given by Brahmagupta in his work *Brahmasphuṭasiddhānta* (Verse 12.55), who referred to the use of the property for multiplication as *khaṇḍa-guṇanam* (multiplication by parts). His verse states, "The multiplier is broken up into two or more parts whose sum is equal to it; the multiplicand is then multiplied by each of these and the results added". That is, if there are two parts, then using letter symbols this is equivalent to the identity $(a + b)\,c = ac + bc$. In the next verse (Verse 12.56), Brahmagupta further describes a method for doing fast multiplication using this distributive property, which we explore further in the next section.

Figure it Out

1. Observe the multiplication grid below. Each number inside the grid is formed by multiplying two numbers. If the middle number of a 3 × 3 frame is given by the expression pq, as shown in the figure, write the expressions for the other numbers in the grid.

x	1	2	3	4	5	6	7	8	9	10
1	1	2	3	4	5	6	7	8	9	10
2	2	4	6	8	10	12	14	16	18	20
3	3	6	9	12	15	18	21	24	27	30
4	4	8	12	16	20	24	28	32	36	40
5	5	10	15	20	25	30	35	40	45	50
6	6	12	18	24	30	36	42	48	54	60
7	7	14	21	28	35	42	49	56	63	70
8	8	16	24	32	40	48	56	64	72	80
9	9	18	27	36	45	54	63	72	81	90
10	10	20	30	40	50	60	70	80	90	100

3 × 5	3 × 6	3 × 7
4 × 5	4 × 6	4 × 7
5 × 5	5 × 6	5 × 7

	pq	

2. Expand the following products.

(i) $(3 + u)(v - 3)$	(ii) $\frac{2}{3}(15 + 6a)$
(iii) $(10a + b)(10c + d)$	(iv) $(3 - x)(x - 6)$
(v) $(-5a + b)(c + d)$	(vi) $(5 + z)(y + 9)$

3. Find 3 examples where the product of two numbers remains unchanged when one of them is increased by 2 and the other is decreased by 4. Math Talk

4. Expand (i) $(a + ab - 3b^2)(4 + b)$, and (ii) $(4y + 7)(y + 11z - 3)$.

5. Expand (i) $(a - b)(a + b)$, (ii) $(a - b)(a^2 + ab + b^2)$ and (iii) $(a - b)(a^3 + a^2b + ab^2 + b^3)$, Do you see a pattern? What would be the next identity in the pattern that you see? Can you check it by expanding?

Fast Multiplications Using the Distributive Property

The distributive property can be used to come up with quick methods of multiplication when certain types of numbers are multiplied.

When one of the numbers is 11, 101, 1001, ...

? Use the following multiplications to find the product of a number with 11 in a single step.

(a) 3874×11 (b) 5678×11

Let us take the first multiplication.

$$3874 \times 11 = 3874\,(10 + 1) = 38740 + 3874$$

$$\begin{array}{r} 38740 \\ +\quad 3874 \\ \hline \\ \hline \end{array}$$

Notice how the digits are getting added.

Let us take a 4-digit number *dcba*, that is, the number that has *d* in the thousands place, *c* in the hundreds place, b in the tens place and *a* in the units place.

$$dcba \times (10 + 1) = dcba \times 10 + dcba.$$

This becomes

$$\begin{array}{cccccc} & d & c & b & a & o \\ + & & d & c & b & a \\ \hline & d & (c + d) & (b + c) & (a + b) & \text{a} \end{array}$$

This can be used to obtain the product in one line.

Step 1	Step 2	Step 3
	①	①①
387④ × 11	3874 × 11	3874 × 11
4	14	614

Step 4	Step 5
①①①	①①①
3874 × 11	③874 × 11
2614	42614

? Describe a general rule to multiply a number (of any number of digits) by 11 and write the product in one line.

Evaluate (i) 94 × 11, (ii) 495 × 11, (iii) 3279 × 11, (iv) 4791256 × 11.

Math Talk

? Can we come up with a similar rule for multiplying a number by 101?

? Multiply 3874 by 101.

Let us take a 4-digit number *dcba*.

$$dcba \times 101 = dcba \times (100 + 1) = dcba \times 100 + dcba.$$

This becomes

	d	*c*	*b*	*a*	*o*	*o*
+			*d*	*c*	*b*	*a*
	d	*c*	(*b* + *d*)	(*a* + *c*)	*b*	*a*

? Use this to multiply 3874 × 101 in one line.

? What could be a general rule to multiply a number by 101 and write the product in one line? Extend this rule for multiplication by 1001, 10001, ...

? Use this to find (i) 89 × 101, (ii) 949 × 101, (iii) 265831 × 1001, (iv) 1111 × 1001, (v) 9734 × 99 and (vi) 23478 × 999.

Math Talk

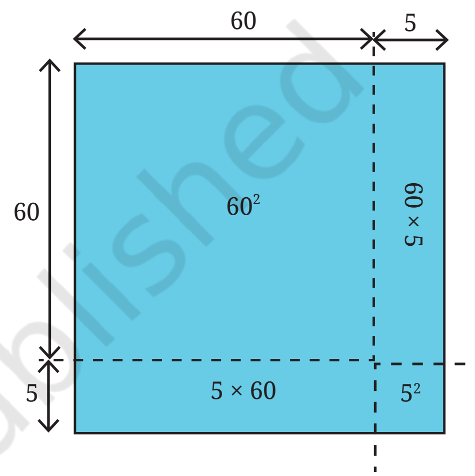
Such methods of applying the distributive property to easily multiply two numbers were discussed extensively in the ancient mathematical works of Brahmagupta (628 CE), Sridharacharya (750 CE) and Bhaskaracharya (Lilavati, 1150 CE). In his work *Brahmasphuṭasiddhānta* (Verse 12.56), Brahmagupta refers to such methods for fast multiplication using the distributive property as **ista-gunana.**

6.2 Special Cases of the Distributive Property

Square of the Sum/Difference of Two Numbers

? The area of a square of sidelength 60 units is 3600 sq. units (60^2) and that of a square of sidelength 5 units is 25 sq. units (5^2). Can we use this to find the area of a square of sidelength 65 units?

A square of sidelength 65 can be split into 4 regions as shown in the figure—a square of sidelength 60, a square of sidelength 5, and two rectangles of sidelengths 60 and 5. The area of the square of sidelength 65 is the sum of the areas of all its constituent parts. Can you find the areas of the four parts in the figure above?



We get

$65^2 = (60 + 5)^2 = 60^2 + 5^2 + 2 \times (60 \times 5)$.

$= 3600 + 25 + 600 = 4225$ sq. units.

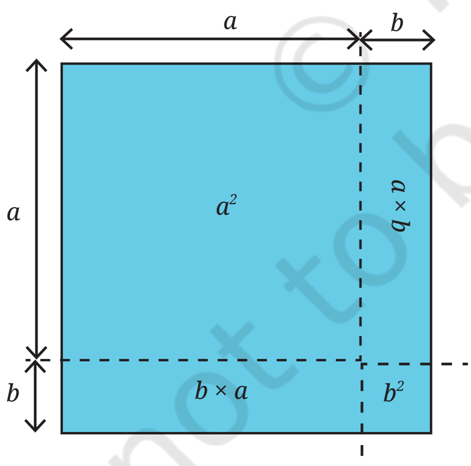
Let us multiply $(60 + 5) \times (60 + 5)$ using the distributive property.

$$(60 + 5) \times (60 + 5) = 60 \times 60 + 5 \times 60 + 60 \times 5 + 5 \times 5$$

$$= 60^2 + 2 \times (60 \times 5) + 5^2.$$

? What if we write 65^2 as $(30 + 35)^2$ or $(52 + 13)^2$? Draw the figures and check the area that you get.

Let us look at the general expression for the square of sum of two numbers, $(a + b)^2$.



Using the distributive property, $(a+b)^2$ can be expanded as

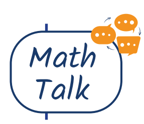
$$(a+b) \times (a+b) = a \times a + a \times b + b \times a + b \times b$$

$$= a^2 + 2ab + b^2,$$

as we had already seen in Example 2.

Identity 1A $(a+b)^2 = a^2 + 2ab + b^2$

? If a and b are any two integers, is $(a + b)^2$ always greater than $a^2 + b^2$? If not, when is it greater?



? Use Identity 1A to find the values of 104^2, 37^2. (**Hint:** Decompose 104 and 37 into sums or differences of numbers whose squares are easy to compute.)

? Use Identity 1A to write the expressions for the following.

(i) $(m + 3)^2$ (ii) $(6 + p)^2$

? Expand $(6x + 5)^2$.

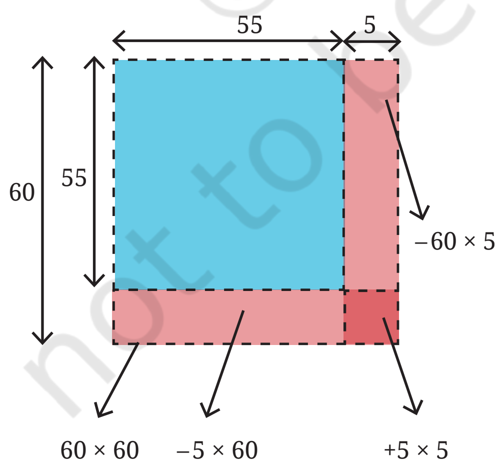
Using the Distributive Property	Using the Identity
$(6x + 5)^2 = (6x + 5)(6x + 5)$ $= (6x \times 6x) + (5 \times 6x) + (6x \times 5) + 5 \times 5$ $= (6x)^2 + 2(6x \times 5) + 5^2$ $= 36x^2 + 60x + 25$.	$(6x + 5)^2 = (6x)^2 + 5^2 + 2 \times (6x \times 5)$ $= 36x^2 + 25 + 60x$.

If you have difficulty remembering or using the general rule, you can just apply the distributive property to multiply and get the desired result.

? Expand $(3j + 2k)^2$ using both the identity and by applying the distributive property.

? Can we use 60^2 (=3600) and 5^2 (=25) to find the value of $(60 – 5)^2$ or 55^2? Let us approach this through geometry by drawing a square of side length 55 sitting inside a square of sidelength 60.

Area of a square of sidelength 55 is $(60 – 5)^2 = 55^2$.



We can get the area of the square of sidelength 55 by taking the area of the square of sidelength 60 and removing the areas of the two rectangles of sidelengths 60 and 5, i.e., $60^2 – (60 \times 5) – (5 \times 60)$. By doing this, we remove the area of the small square of sidelength 5 twice. What can we do with this expression to get the actual area?

We can add back the area of the square of sidelength 5 to this expression. That way, we are only subtracting this area once.

So,

$$(60 - 5)^2 = 60^2 - (60 \times 5) - (5 \times 60) + 5^2$$
$$= 3600 - 300 - 300 + 25$$
$$= 3025.$$

The area of the square of sidelength 55 is 3025 sq. units.

We have seen what $(a + b)^2$ gives when expanded. What is the expansion of $(a - b)^2$?

Using the distributive property,

$$(a - b)^2 = (a - b) \times (a - b)$$
$$= (a)^2 - ba - ab + (b)^2$$
$$= a^2 - 2ab + b^2.$$

? We can also use the expansion of $(a + b)^2$ to find the expansion of $(a - b)^2$. Think how.

Hint: $(a - b)^2 = (a + (-b))^2$.

We can now directly use the expansion of $(a + b)^2$.

$$(a + (-b))^2 = (a)^2 + (-b)^2 + 2 \times (a) \times (-b)$$

Identity 1B $(a - b)^2 = a^2 + b^2 - 2ab$

? Find the general expansion of $(a - b)^2$ using geometry, as we did for 55^2.

? Use the identity $(a - b)^2$ to find the values of (a) 99^2 and (b) 58^2.

? Expand the following using both Identity 1B and by applying the distributive property

(i) $(b - 6)^2$ (ii) $(-2a + 3)^2$ (iii) $(7y - \frac{3}{4z})^2$

Investigating Patterns

Pattern 1

Look at the following pattern.

$$2\,(2^2 + 1^2) = 3^2 + 1^2$$
$$2\,(3^2 + 1^2) = 4^2 + 2^2$$
$$2\,(6^2 + 5^2) = 11^2 + 1^2$$
$$2\,(5^2 + 3^2) = 8^2 + 2^2.$$

? Take a pair of natural numbers. Calculate the sum of their squares. Can you write twice this sum as a sum of two squares?

Try this with other pairs of numbers. Have you figured out a pattern?

Notice that $2\,(5^2 + 6^2) = (6 + 5)^2 + (6 - 5)^2$.

? Do the identities below help in explaining the observed pattern?

$$(a + b)^2 = a^2 + 2ab + b^2$$

$$(a - b)^2 = a^2 - 2ab + b^2$$

$$(a + b)^2 + (a - b)^2 = (a^2 + 2ab + b^2) + (a^2 - 2ab + b^2)$$

Adding the like terms $a^2 + a^2 = 2a^2$, $b^2 + b^2 = 2b^2$ and $2ab - 2ab = 0$, we get

$$2\,(a^2 + b^2) = (a + b)^2 + (a - b)^2.$$

Pattern 2

? Here is a related pattern. Try to describe the pattern using algebra to determine if the pattern always holds.

$$9 \times 9 - 1 \times 1 = 10 \times 8$$

$$8 \times 8 - 6 \times 6 = 14 \times 2$$

$$7 \times 7 - 2 \times 2 = 9 \times 5$$

$$10 \times 10 - 4 \times 4 = 14 \times 6$$

The pattern here appears to be $a^2 - b^2 = (a + b) \times (a - b)$.

Is this a true identity? Using the distributive property, we get

$$(a + b) \times (a - b) = a^2 - ab + ba - a^2.$$

Adding the like terms, $ab + (-ab) = 0$, we see that indeed

Identity 1C $(a + b) \times (a - b) = a^2 - b^2$.

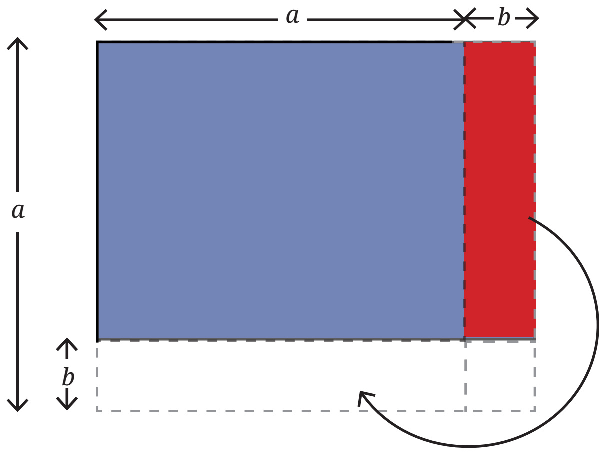
You had seen this identity earlier in Figure it Out 5 (i).

? Use Identity 1C to calculate 98×102, and 45×55.

? Show that $(a + b) \times (a - b) = a^2 - b^2$ geometrically.

Try This

Hint:



Sridharacharya (750 CE) gave an interesting method to quickly compute the squares of numbers using Identity 1C! Consider the following modified form of this identity —

$$a^2 = (a + b)(a - b) + b^2$$

Why is this identity true?

Now, for example, 31^2 can be found by taking $a = 31$ and $b = 1$.

$$31^2 = (31 + 1)(31 - 1) + 1^2$$
$$= 32 \times 30 + 1$$
$$= 961.$$

197^2 can be found by taking $a = 197$, and $b = 3$.

$$197^2 = (197 + 3)(197 - 3) + 3^2$$
$$= 200 \times 194 + 9$$
$$= 38809.$$

Figure it Out

1. Which is greater: $(a - b)^2$ or $(b - a)^2$? Justify your answer.
2. Express 100 as the difference of two squares.
3. Find 406^2, 72^2, 145^2, 1097^2, and 124^2 using the identities you have learnt so far.
4. Do Patterns 1 and 2 hold only for counting numbers? Do they hold for negative integers as well? What about fractions? Justify your answer.

Math Talk

6.3 Mind the Mistake, Mend the Mistake

We have expanded and simplified some algebraic expressions below to their simplest forms.

(i) Check each of the simplifications and see if there is a mistake.

(ii) If there is a mistake, try to explain what could have gone wrong.

(iii) Then write the correct expression.

1.
$-3p(-5p + 2q)$
$= -3p + 5p - 2q$
$= p - 2q$

2.
$2(x - 1) + 3(x + 4)$
$= 2x - 1 + 3x + 4$
$= 5x + 3$

3.
$y + 2(y + 2)$
$= (y + 2)^2$
$= y^2 + 4y + 4$

4.
$(5m + 6n)^2$
$= 25m^2 + 36n^2$

5.
$(-q + 2)^2$
$= q^2 - 4q + 4$

6.
$3a(2b \times 3c)$
$= 6ab \times 9ac$
$= 54a2bc$

7.
$\frac{1}{2}(10s - 6) + 3$
$= 5s - 3 + 3$
$= 5s$

8.
$5w^2 + 6w$
$= 11w^2$

9.
$2a^3 + 3a^3 + 6a^2b + 6ab^2$
$= 5a^3 + 12a^2b^2$

10.
$(x + 2)(x + 5)$
$= (x + 2)x + (x + 2)5$
$= x^2 + 2x + 5x + 10$
$= x^2 + 7x + 10$

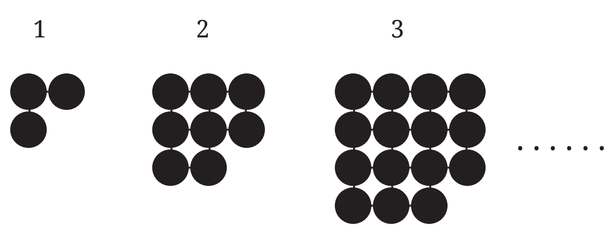
11.
$(a + 2)(b + 4)$
$= ab + 8$

12.
$ab^2 + a^2b + a^2b^2$
$= ab(a + b + ab)$

6.4 This Way or That Way, All Ways Lead to the Bay

Math Talk

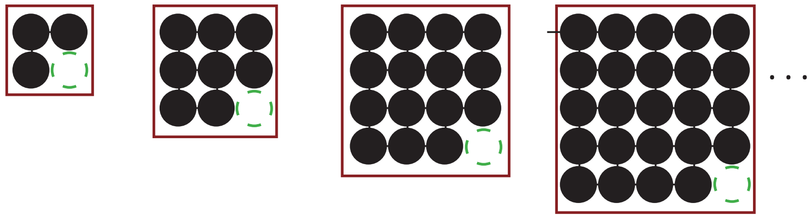
Observe the pattern in the figure below. Draw the next figure in the sequence. How many circles does it have? How many total circles are there in Step 10? Write an expression for the number of circles in Step *k*.



There are many ways of interpreting this pattern. Here are some possibilities:

Method 1

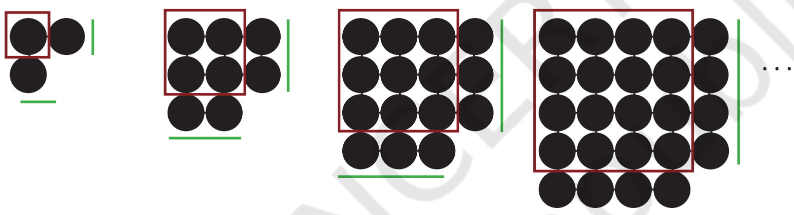
Step 1	Step 2	Step 3	Step 4	. . .	Step k



$2^2 - 1$ $= (1 + 1)^2 - 1$	$3^2 - 1$ $= (2 + 1)^2 - 1$	$4^2 - 1$ $= (3 + 1)^2 - 1$	$5^2 - 1$ $= (4 + 1)^2 - 1$	. . .	$(k + 1)^2 - 1$

Method 2

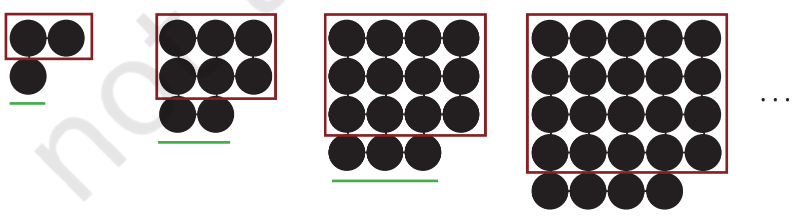
Step 1	Step 2	Step 3	Step 4	. . .	Step k



$1 + 2 \times 1$ $= 1^2 + 2 \times 1$	$2^2 + 2 \times 2$ $= 2^2 + 2 \times 2$	$3^2 + 2 \times 3$ $= 3^2 + 2 \times 3$	$4^2 + 2 \times 4$ $= 4^2 + 2 \times 4$	. . .	$k^2 + 2 \times k$

Method 3

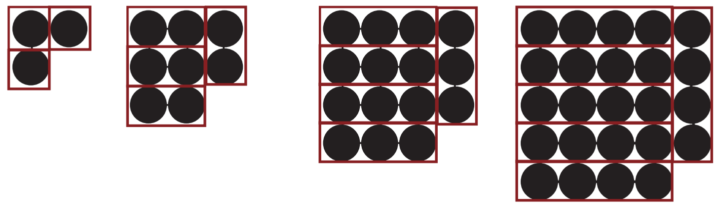
Step 1	Step 2	Step 3	Step 4	. . .	Step k



$1 \times 2 + 1$ $= 1 \times (1 + 1) + 1$	$2 \times 3 + 2$ $= 2 \times (2 + 1) + 2$	$3 \times 4 + 3$ $= 3 \times (3 + 1) + 3$	$4 \times 5 + 4$ $= 4 \times (4 + 1) + 4$	. . .	$k \times (k + 1) + k$

Method 4

Step **1**   Step **2**   Step **3**   Step **4**   . . .   Step *k*



Step 1	Step 2	Step 3	Step 4		Step k
1×3	2×4	3×5	4×6		
$= \mathbf{1} \times (\mathbf{1} + 2)$	$= \mathbf{2} \times (\mathbf{2} + 2)$	$= \mathbf{3} \times (\mathbf{3} + 2)$	$= \mathbf{4} \times (\mathbf{4} + 2)$	. . .	$k \times (k+2)$

Does your method match any of these, or is it different? Each expression that we have identified appears different, but are they really different? Since they describe the same pattern, they should all be the same. Let us simplify each expression and find out.

$$(k + 1)^2 - 1 = k^2 + 1 + 2k - 1 = k^2 + 2k$$

$$k^2 + 2 \times k = k^2 + 2k$$

$$k \times (k + 1) + k = k^2 + k + k = k^2 + 2k$$

$$k \times (k + 2) = k^2 + 2k$$

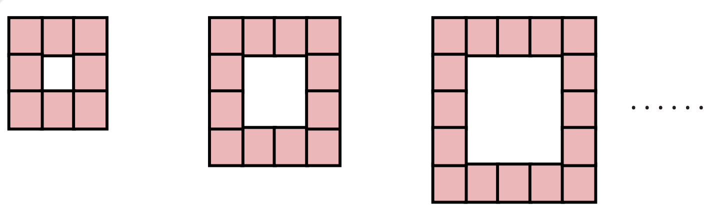
When carried out correctly, all methods lead to the same answer; $k^2 + 2k$. The expression $k^2 + 2k$ gives the number of circles at Step k of this pattern.

In Mathematics, there are often multiple ways of looking at a pattern, and different ways of approaching and solving the same problem. Finding such ways often requires a great deal of creativity and imagination! While one or two of the ways might be your favourite(s), it can be amusing and enriching to explore other ways as well.

? Use this formula to find the number of circles in Step 15.

? Consider the pattern made of square tiles in the picture below.

1   2   3



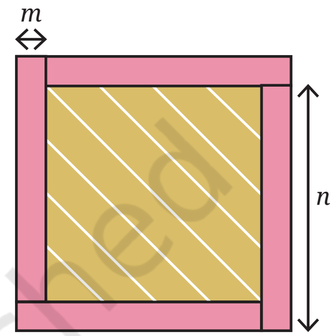
? How many square tiles are there in each figure?

? How many are there in Step 4 of the sequence? What about Step 10?

? Write an algebraic expression for the number of tiles in Step n. Share your methods with the class. Can you find more than one method to arrive at the answer?

Math Talk

? Find the area of the (interior) shaded region in the figure below. All four rectangles have the same dimensions.



Tadang's method:

The total region is a square of side $(m + n)$ with an area $(m + n)^2$.

Subtracting the area of four rectangles from the total area will give the area of the interior shaded region. That is, $(m + n)^2 - 4mn$.

Yusuf's method:

The shaded region is a square with sidelength $(n - m)$. So, its area is $(n - m)^2$.

? By expanding both expressions, check that $(m + n)^2 - 4mn = (n - m)^2$.

? Find out the area of the region with slanting lines in the figure. All three rectangles have the same dimensions (Fig. 1).

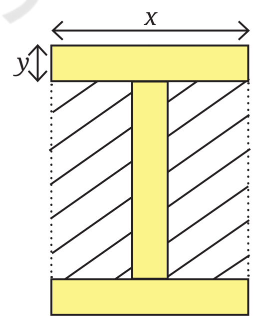


Fig. 1

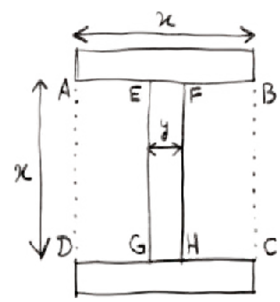
Anusha's method:

Required area = Area (ABCD) – Area (EFGH)

Area of ABCD = x^2.

Area of EFGH = xy.

Required area = $x^2 - xy$.



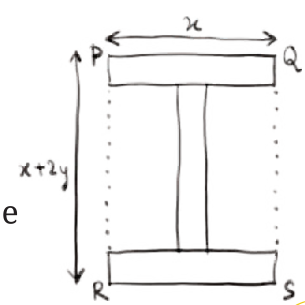
Vaishnavi's method:

$$QS = y + x + y$$
$$= x + 2y.$$

Area of PQSR = $x\,(x + 2y)$

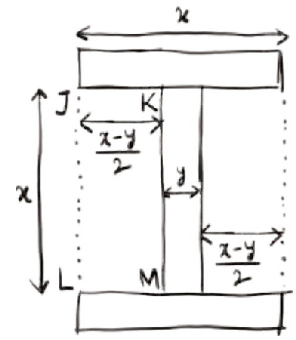
Required area = Area of PQSR – (area of the three rectangles)

$$= x\,(x + 2y) - 3xy.$$



Aditya's method:

The required area is 2 times the area of JKLM.



$$JK = \frac{x-y}{2}, \; KM = x$$

Area (JKML) = $x\left(\frac{x-y}{2}\right)$

Required area = 2 × Area of JKML

$= 2x\left(\frac{x-y}{2}\right)$

$= x(x-y)$.

By expanding the expressions, verify that all three expressions are equivalent. If $x = 8$ and $y = 3$, find the area of the shaded region.

Write an expression for the area of the dashed region in the figure below. Use more than one method to arrive at the answer. Substitute p = 6, $r = 3.5$, and $s = 9$, and calculate the area.

Math Talk

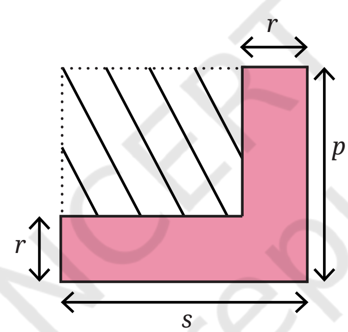


Figure it Out

1. Compute these products using the suggested identity.
   (i) 46^2 using Identity 1A for $(a + b)^2$
   (ii) 397 × 403 using Identity 1C for $(a + b)(a - b)$
   (iii) 91^2 using Identity 1B for $(a - b)^2$
   (iv) 43 × 45 using Identity 1C for $(a + b)(a - b)$
2. Use either a suitable identity or the distributive property to find each of the following products.
   (i) $(p - 1)(p + 11)$
   (ii) $(3a - 9b)(3a + 9b)$
   (iii) $-(2y + 5)(3y + 4)$
   (iv) $(6x + 5y)^2$
   (v) $(2x - \frac{1}{2})^2$
   (vi) $(7p) \times (3r) \times (p + 2)$

3. For each statement identify the appropriate algebraic expression(s).

   (i) Two more than a square number.

   $2 + s$ $\quad (s + 2)^2 \quad s^2 + 2 \quad s^2 + 4 \quad 2s^2 \quad 2^2s$

   (ii) The sum of the squares of two consecutive numbers

   $m^2 + n^2 \quad (m + n)^2 \quad m^2 + 1 \quad m^2 + (m + 1)^2$

   $m^2 + (m - 1)^2 \quad (m + (m + 1))^2 \quad (2m)^2 + (2m + 1)^2$

4. Consider any 2 by 2 square of numbers in a calendar, as shown in the figure.

February

Su	M	Tu	W	Th	F	Sa
						1
2	3	4	5	6	7	8
9	10	11	12	13	14	15
16	17	18	19	20	21	22
23	24	25	26	27	28	

Find products of numbers lying along each diagonal — $4 \times 12 = 48$, $5 \times 11 = 55$. Do this for the other 2 by 2 squares. What do you observe about the diagonal products? Explain why this happens.

Math Talk

Hint: Label the numbers in each 2 by 2 square as

a	$(a + 1)$
$a + 7$	$(a + 8)$

5. Verify which of the following statements are true.

   (i) $(k + 1)(k + 2) - (k + 3)$ is always 2.

   (ii) $(2q + 1)(2q - 3)$ is a multiple of 4.

   (iii) Squares of even numbers are multiples of 4, and squares of odd numbers are 1 more than multiples of 8.

   (iv) $(6n + 2)^2 - (4n + 3)^2$ is 5 less than a square number.

6. A number leaves a remainder of 3 when divided by 7, and another number leaves a remainder of 5 when divided by 7. What is the remainder when their sum, difference, and product are divided by 7?

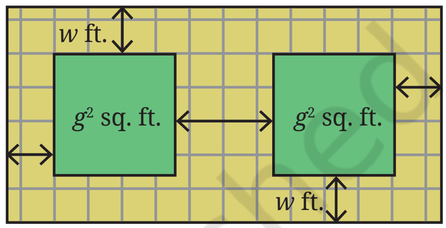
7. Choose three consecutive numbers, square the middle one, and subtract the product of the other two. Repeat the same with other

sets of numbers. What pattern do you notice? How do we write this as an algebraic equation? Expand both sides of the equation to check that it is a true identity.

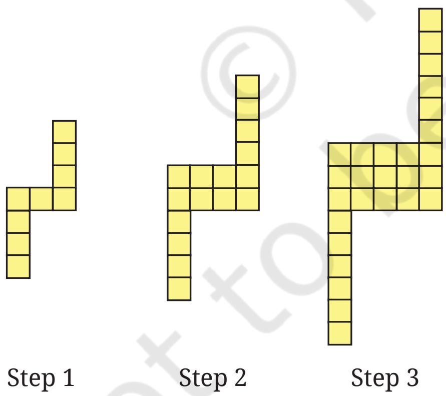
8. What is the algebraic expression describing the following steps—add any two numbers. Multiply this by half of the sum of the two numbers? Prove that this result will be half of the square of the sum of the two numbers.

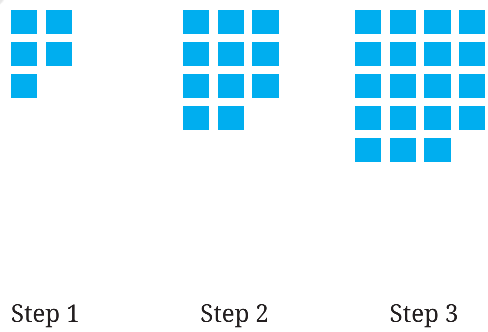
9. Which is larger? Find out without fully computing the product.
   (i) 14 × 26 or 16 × 24
   (ii) 25 × 75 or 26 × 74

10. A tiny park is coming up in Dhauli. The plan is shown in the figure. The two square plots, each of area g^2 sq. ft., will have a green cover. All the remaining area is a walking path w ft. wide that needs to be tiled. Write an expression for the area that needs to be tiled.



11. For each pattern shown below,
   (i) Draw the next figure in the sequence.
   (ii) How many basic units are there in Step 10?
   (iii) Write an expression to describe the number of basic units in Step y.





SUMMARY

- We extended the distributive property to find the product of two expressions each of which has two terms. The general form for the same is $(a + b) \times (c + d) = ac + ad + bc + bd$.
- We saw some special cases of this identity.
  - $(a + b)^2 = a^2 + 2ab + b^2$
  - $(a - b)^2 = a^2 - 2ab + b^2$
  - $(a + b)(a - b) = a^2 - b^2$.
- We considered different patterns, and explored how to understand them using algebra. We saw that, often, there are multiple ways to solve a problem and arrive at the same correct answer. Finding different methods to approach and solve the same problem is a creative process.

Coin Conjoin

Arrange 10 coins in a triangle as shown in the figure below on the left. The task is to turn the triangle upside down by moving one coin at a time. How many moves are needed? What is the minimum number of moves?

A triangle of 3 coins can be inverted (turned upside down) with a single move, and a triangle of 6 coins can be inverted by moving 2 coins.

The 10-coin triangle can be flipped with just 3 moves; did you figure out how? Find out the minimum possible moves needed to flip the next bigger triangle having 15 coins. Try the same for bigger triangular numbers.

Is there a simple way to calculate the minimum number of coin moves needed for any such triangular arrangement?

7 PROPORTIONAL REASONING-1

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7.1 Observing Similarity in Change

We are all familiar with digital images. We often change the size and orientation of these images to suit our needs. Observe the set of images below—

Image A

Image B

Image C

Image D

Image E

We can see that all the images are of different sizes.

? Which images look similar and which ones look different?

Images (A, C, and D) look similar, even though they have different sizes.

Do images B and E look like the other three images?

No, they are slightly distorted. The tiger appears elongated in B, and compressed and fatter in E!

Why?

You may notice that images A, C, and D are rectangular, but E is square. Maybe that is why E looks different. But B is also a rectangle! Why does it look different from the other rectangular images?

Math Talk

Can we observe any pattern to answer this question? Perhaps by measuring the rectangles?

Image	Width (in mm)	Height (in mm)
Image A	60	40
Image B	40	20
Image C	30	20
Image D	90	60
Image E	60	60

What makes images A, C, and D appear similar, and B and E different?

When we compare image A with C, we notice that the width of C is half that of A. The height is also half of A. Both the width and height have **changed by the same factor** (through multiplication), $\frac{1}{2}$ in this case. Since the widths and heights have changed by the same factor, the **images look similar**.

When we compare image A with image B, we notice that the width of B is 20 millimetre (mm) less than that of A. The height too is 20 mm less than the height of A. Even though the difference (through subtraction) is the same, the images look different. Have the width and height changed by the same factor? The height of B is half the height of A. But the width of B is not half the width of A. Since the width and height have not changed by the same factor, the images look different.

Can you check by what factors the width and height of image D change as compared to image A? Are the factors the same?

Images A, C, and D look similar because their widths and heights have changed by the same factor. We say that the changes to their widths and heights are **proportional.**

7.2 Ratios

We use the notion of a **ratio** to represent such proportional relationships in mathematics.

We can say that the ratio of width to height of image A is

60 : 40.

The numbers 60 and 40 are called the **terms** of the ratio.

The ratio of width to height of image C is 30 : 20, and that of image D is 90 : 60.

> In a ratio of the form $a : b$, we can say that for every 'a' units of the first quantity, there are 'b' units of the second quantity.

So, in image A, we can say that for every 60 mm of width, there are 40 mm of height.

We can say that the ratios of width to height of images A, C, and D are proportional because the terms of these ratios change by the same factor. Let us see how.

Image A — 60 : 40

Multiplying both the terms by $\frac{1}{2}$, we get

$$60 \times \frac{1}{2} : 40 \times \frac{1}{2}$$

which is 30 : 20, the ratio of width to height in image C.

? By what factor should we multiply the ratio 60 : 40 (image A) to get 90 : 60 (image D)?

A more systematic way to compare whether the ratios are proportional is to reduce them to their **simplest form** and see if these simplest forms are the same.

7.3 Ratios in their Simplest Form

We can reduce ratios to their simplest form by dividing the terms by their HCF.

In image A, the terms are 60 and 40. What is the HCF of 60 and 40? It is 20. Dividing the terms by 20, we get the ratio of image A to be 3 : 2 in its simplest form.

The ratio of image D is 90 : 60. Dividing both terms by 30 (HCF of 90 and 60), we get the simplest form to be 3 : 2. So the ratios of images A and D are proportional as well.

What is the simplest form of the ratios of images B and E?

The ratio of image B is 40 : 20; in its simplest form, it is 2 : 1.

The ratio of image E is 60 : 60; in its simplest form, it is 1 : 1.

These ratios are not the same as 3 : 2. So, we can say that the ratios of width to height of images B and E are not proportional to the ratios of images A, C, and D.

When two ratios are the same in their simplest forms, we say that the ratios are in **proportion**, or that the ratios are **proportional**. We use the '::' symbol to indicate that they are proportional.
So $a : b :: c : d$ indicates that the ratios $a : b$ and $c : d$ are proportional.

Thus,

$$60 : 40 :: 30 : 20 \text{ and } 60 : 40 :: 90 : 60.$$

7.4 Problem Solving with Proportional Reasoning

Example 1: Are the ratios 3 : 4 and 72 : 96 proportional?

3 : 4 is already in its simplest form.

To find the simplest form of 72 : 96, we need to divide both terms by their HCF.

What is the HCF of 72 and 96?

The HCF of 72 and 96 is 24. Dividing both terms by 24, we get 3 : 4. Since both ratios in their simplest form are the same, they are proportional.

Example 2: Kesang wanted to make lemonade for a celebration. She made 6 glasses of lemonade in a vessel and added 10 spoons of sugar to the drink. Her father expected more people to join the celebration. So he asked her to make 18 more glasses of lemonade.

To make the lemonade with the same sweetness, how many spoons of sugar should she add?

To maintain the same sweetness, the ratio of the number of glasses of lemonade to the number of spoons of sugar should be proportional. For 6 glasses of lemonade, she added 10 spoons of sugar.

The ratio of glasses of lemonade to spoons of sugar is 6 : 10. If she needs to make 18 more glasses of lemonade, how many spoons of sugar should she use? We can model this problem as —

$$6 : 10 :: 18 : ?$$

We know that each term in the ratio must change by the same factor, for the ratios to be proportional.

How can we find the factor of change in the ratio?

The first term has increased from 6 to 18. To find the factor of change, we can divide 18 by 6 to get 3.

The second term should also change by the same factor. When 10 increases by a factor of 3, it becomes 30. Thus,

$$6 : 10 :: 18 : 30.$$

So, she should use 30 spoons of sugar to make 18 glasses of lemonade with the same sweetness as earlier.

? **Example 3:** Nitin and Hari were constructing a compound wall around their house. Nitin was building the longer side, 60 ft in length, and Hari was building the shorter side, 40 ft in length. Nitin used 3 bags of cement but Hari used only 2 bags of cement. Nitin was worried that the wall Hari built would not be as strong as the wall he built because she used less cement.

? Is Nitin correct in his thinking?

In Nitin and Hari's case, we should compare the ratio of the length of the wall to the bags of cement used by each of them and see whether they are proportional.

The ratio in Nitin's case is 60 : 3, i.e., 20 : 1 (in its simplest form).
The ratio in Hari's case is 40 : 2, i.e., 20 : 1 (in its simplest form).

Since both ratios are proportional, the walls are equally strong. Nitin should not worry!

? **Example 4:** In my school, there are 5 teachers and 170 students. The ratio of teachers to students in my school is 5 : 170. Count the number of teachers and students in your school. What is the ratio of teachers to students in your school? Write it below.

______ : ______

? Is the teacher-to-student ratio in your school proportional to the one in my school?

? **Example 5:** Measure the width and height (to the nearest cm) of the blackboard in your classroom. What is the ratio of width to height of the blackboard?

______ : ______

? Can you draw a rectangle in your notebook whose width and height are proportional to the ratio of the blackboard?

? Compare the rectangle you have drawn to those drawn by your classmates. Do they all look the same?

Math Talk

Note to the Teacher: Give more such examples that students can relate to and ask them to give reasons why they think they are correct. Engaging with these problems and finding solutions through a process of **proportional reasoning** should go along with learning procedures and methods to solve the problems.

Example 6: When Neelima was 3 years old, her mother's age was 10 times her age. What is the ratio of Neelima's age to her mother's age? What would be the ratio of their ages when Neelima is 12 years old? Would it remain the same?

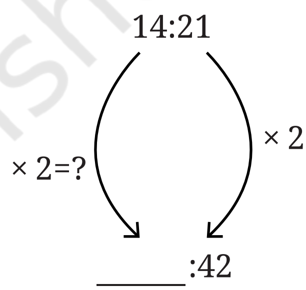
The ratio of Neelima's age to her mother's age when Neelima is 3 years old is 3 : 30 (her mother's age is 10 times Neelima's age). In the simplest form, it is 1 : 10.

When Neelima is 12 years old (i.e., 9 years later), the ratio of their ages will be 12 : 39 (9 years later, her mother would be 39 years old). In the simplest form, it is 4 : 13.

When we add (or subtract) the same number from the terms of a ratio, the ratio changes and is not necessarily proportional to the original ratio.

Example 7: Fill in the missing numbers for the following ratios that are proportional to 14 : 21.

_____ : 42 6 : _____ 2 : _____



In the first ratio, we don't know the first term. But the second term is 42. It is 2 times the second term of the ratio 14 : 21. So, the first term should also be 2 times 14 (the first term). Hence the proportional ratio is 28 : 42.

For the second ratio, the first term is 6.

What factor should we multiply 14 by to get 6? Can it be an integer? Or should it be a fraction?

We can model this as $14y = 6$. So, $y = \frac{6}{14} = \frac{3}{7}$.

So, we need to multiply 21 (the second term of 14 : 21) also by the same factor $\frac{3}{7}$.

$21 \times \frac{3}{7}$ is 9. So, the ratio is 6 : 9.

In the third ratio, the first term is 2.

We can see that when we divide 14 (the first term of 14 : 21) by 7 (HCF of 14 and 21) we get 2.

If we divide 21 also by 7, we get 3. So, the ratio is 2 : 3.

Filter Coffee!

Filter coffee is a beverage made by mixing coffee decoction with milk. Manjunath usually mixes 15 mL of coffee decoction with 35 mL of milk to make one cup of filter coffee in his coffee shop.

In this case, we can say that the ratio of coffee decoction to milk is 15 : 35.

If customers want 'stronger' filter coffee, Manjunath mixes 20 mL of decoction with 30 mL of milk. The ratio here is 20 : 30.

? Why is this coffee stronger?

Math Talk

And when they want 'lighter' filter coffee, he mixes 10 mL of coffee and 40 mL of milk, making the ratio 10 : 40.

? Why is this coffee lighter?

Math Talk

The following table shows the different ratios in which Manjunath mixes coffee decoction with milk. Write in the last column if the coffee is stronger or lighter than the regular coffee.

Coffee Decoction (in mL)	Milk (in mL)	Regular/Strong/ Light
300	600	
150	500	
200	400	
24	56	
100	300	

Figure it Out

1. Circle the following statements of proportion that are true.

(i) 4 : 7 :: 12 : 21

(ii) 8 : 3 :: 24 : 6

(iii) 7 : 12 :: 12 : 7

(iv) 21 : 6 :: 35 : 10

(v) 12 : 18 :: 28 : 12

(vi) 24 : 8 :: 9 : 3

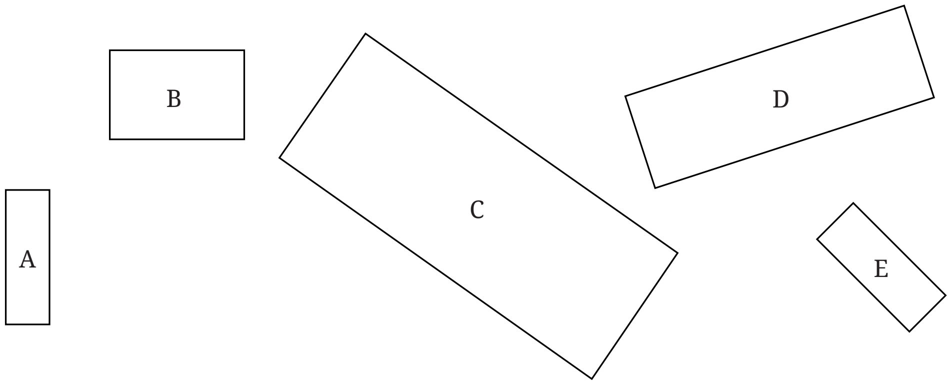
2. Give 3 ratios that are proportional to 4 : 9.

______ : ______   ______ : ______   ______ : ______

3. Fill in the missing numbers for these ratios that are proportional to 18 : 24.

3 : ______   12 : ______   20 : ______   27 : ______

4. Look at the following rectangles. Which rectangles are similar to each other? You can verify this by measuring the width and height using a scale and comparing their ratios.



5. Look at the following rectangle. Can you draw a smaller rectangle and a bigger rectangle with the same width to height ratio in your notebooks? Compare your rectangles with your classmates' drawings.

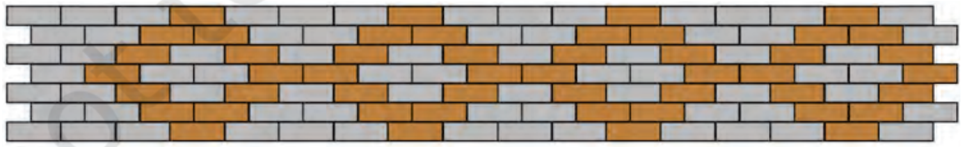
Are all of them the same? If they are different from yours, can you think why? Are they wrong?

Math Talk

6. The following figure shows a small portion of a long brick wall with patterns made using coloured bricks. Each wall continues this pattern throughout the wall. What is the ratio of grey bricks to coloured bricks? Try to give the ratios in their simplest form.

(a)

(b)



7. Let us draw some human figures. Measure your friend's body—the lengths of their head, torso, arms, and legs. Write the ratios as mentioned below—

head : torso

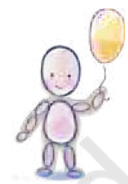
______ : ______

torso : arms

______ : ______

torso : legs

______ : ______



Now, draw a figure with head, torso, arms, and legs with equivalent ratios as above.

Does the drawing look more realistic if the ratios are proportional? Why? Why not?

Math Talk

Note to the Teacher: In all these activities, encourage students to reason why their drawings are proportional.

Trairasika—The Rule of Three

Example 8: For the mid-day meal in a school with 120 students, the cook usually makes 15 kg of rice. On a rainy day, only 80 students came to school. How many kilograms of rice should the cook make so that the food is not wasted?

The ratio of the number of students to the amount of rice needs to be proportional.

So, 120 : 15 :: 80 : ?

What is the factor of change in the first term?

We can find that by dividing the terms $\frac{80}{120} = \frac{2}{3}$.

The number of students is reduced by a factor of $\frac{2}{3}$.

On multiplying the weight of rice by the same factor, we get,

$$15 \times \frac{2}{3} = 10.$$

So, the cook should make 10 kg of rice on that day.

The situation above is a typical example of a problem where we need to use proportional reasoning to find a solution. Four quantities are linked proportionally, out of which three are known and we must find the fourth, unknown, quantity.

To solve such problems, we can model two proportional ratios using algebraic notation as —

$$a : b :: c : d.$$

For these two ratios to be proportional, we know that term c should be a multiple of term a by a factor, say f, and term d should be a multiple of term b by the same factor f. So,

$$c = fa \qquad \text{...(1)}$$

$$d = fb \qquad \text{...(2)}$$

From (1) and (2), we can say that,

$$f = \frac{c}{a} \text{ and } f = \frac{d}{b}.$$

$$\text{Therefore, } \frac{c}{a} = \frac{d}{b}.$$

Multiplying both sides by ab, we get,

$$ab \times \frac{c}{a} = ab \times \frac{d}{b}$$

$$\cancel{a}b \times \frac{c}{\cancel{a}} = a\cancel{b} \times \frac{d}{\cancel{b}}$$

$$bc = ad \text{ or } ad = bc$$

Thus, when $a : b :: c : d$, then $ad = bc$. This is known as cross multiplication of terms.

Since $ad = bc$, we can show that

$$d = \frac{bc}{a}.$$

> Two ratios are proportional if their terms are equal when cross multiplied. The fourth unknown quantity can be found through such cross multiplication.

In ancient India, Āryabhaṭa (199 CE) and others called such problems of proportionality *Rule of Three* problems. There were 3 numbers given—the *pramāṇa* (measure—'a' in our case), the *phala* (fruit—'b' in our case), and the *ichchhā* (requisition—'c' in our case). To find the *ichchhāphala* (yield—'d' in our case), Āryabhaṭa says,

"Multiply the *phala* by the *ichchhā* and divide the resulting product by the *pramāṇa*."

In other words, Āryabhaṭa says,

"pramāṇa : phala :: ichchhā : ichchhāphala," therefore,

pramāṇa × *ichchhāphala* = *phala* × *ichchhā*.

Thus,

$$ichchh\bar{a}phala = \frac{phala \times ichchh\bar{a}}{pram\bar{a}\d{n}a}.$$

Using the cross multiplication method proposed by Āryabhaṭa, ancient Indians solved complex problems that involved proportionality.

Example 9: A car travels 90 km in 150 minutes. If it continues at the same speed, what distance will it cover in 4 hours?

If it continues at the same speed, the ratio of the time taken should be proportional to the ratio of the distance covered.

$$150 : 90 :: 4 : ?$$

Is this the right way to formulate the question?

No, because 150 is in minutes, but 4 is in hours. The second ratio should use the same units for time as the first ratio. Since 4 hours is 240 minutes, the right form is

$$150 : 90 :: 240 : ?$$

How can you find the distance covered in 240 minutes?

Discuss with your classmates and find the answer using different strategies.

Note to the Teacher: Instead of giving one 'method' to solve the problem of the distance, encourage students to reason out the answer through different strategies. They can use their understanding of equivalent fractions and equivalent ratios to find the answer.

We can model this proportion as

$$150 : 90 :: 240 : x.$$

By cross multiplication, we get

$$150 \times x = 240 \times 90$$

Therefore, $x = \frac{240 \times 90}{150}$.

$$= \frac{\overset{48}{\cancel{240}} \times \overset{3}{\cancel{90}}}{\underset{\underset{1}{\cancel{5}}}{\cancel{150}}} = 144.$$

The distance covered by the car in 4 hours is 144 km.

Example 10: A small farmer in Himachal Pradesh sells each 200 g packet of tea for ₹200. A large estate in Meghalaya sells each 1 kg packet of tea for ₹800. Are the weight-to-price ratios in both places proportional? Which tea is more expensive?

The ratio of weight to price of the Himachal tea is 200 : 200.

What is the weight to price ratio of the Meghalaya tea? Is it 1 : 800? This would not be appropriate, because we considered the weight in grams in the case of Himachal. So, the weight to price ratio is 1000 : 800 in Meghalaya after we convert the weight to grams.

To check if the ratios are proportional, we need to see if both ratios are the same in their simplest forms.

The Himachal tea ratio in its simplest form is 1 : 1.

The Meghalaya tea ratio in its simplest form is 5 : 4.

So, the ratios are not proportional.

? Which tea is more expensive? Why?

Note to the Teacher: Encourage a discussion on the more expensive tea, how they came to their conclusions and what the reasons for that tea being more expensive could be.

To answer the question as to which tea is more expensive, we should compare the price of tea for the same weight in both places.

What is the price of 1 kg of tea from Meghalaya? It is ₹800.

In Himachal, if 200 g of tea costs rupees 200, what is the cost of 1 kg of tea?

Let us say that the price of 1 kg of tea is x rupees. 200 g is $\frac{1}{5}$ of 1 kg.

So, $\frac{1}{5} \times x = 200.$

Multiplying both sides by 5, we get

$$\frac{1}{5} \times x \times 5 = 200 \times 5$$

$$\frac{1}{\cancel{5}} \times x \times \cancel{5} = 1000$$

$$x = 1000.$$

So, the cost of 1 kg of tea is ₹800 in Meghalaya and ₹1,000 in Himachal Pradesh.

Therefore, the tea from Himachal Pradesh is more expensive.

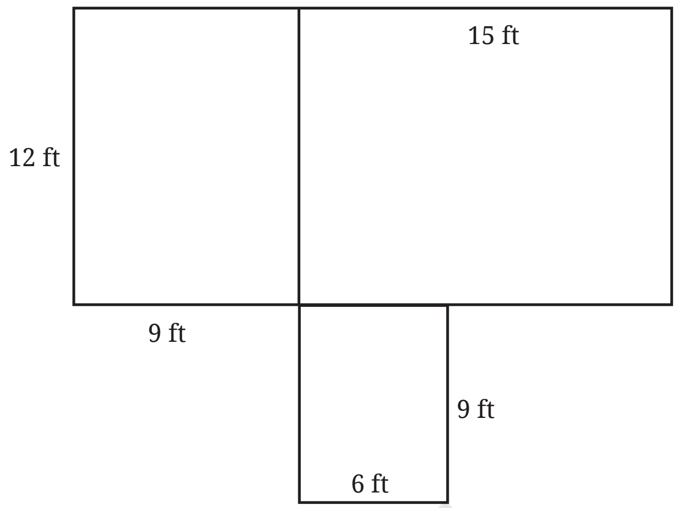
Activity 1: Take your favourite dish. Find out all the ingredients and their respective quantities needed to make the dish for your family. Suppose you are celebrating a festival and you want to invite 15 guests. Find out the quantities of the ingredients required to cook the same dish for them.

? Figure it Out

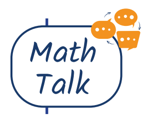
1. The Earth travels approximately 940 million kilometres around the Sun in a year. How many kilometres will it travel in a week?

2. A mason is building a house in the shape shown in the diagram. He needs to construct both the outer walls and the inner wall that

separates two rooms. To build a wall of 10-feet, he requires approximately 1450 bricks. How many bricks would he need to build the house? Assume all walls are of the same height and thickness.



? Puneeth's father went from Lucknow to Kanpur in 2 hours by riding his motorcycle at a speed of 50 km/h. If he drives at 75 km/h, how long will it take him to reach Kanpur? Can we form this problem as a proportion—



$$50 : 2 :: 75 : \_\_$$

Would it take Puneeth's father more time or less time to reach Kanpur? Think about it.

Even though this problem looks similar to the previous problems, it cannot be solved using the Rule of Three!

The time of travel would actually decrease when the speed increases. So this problem cannot be modelled as 50 : 2 :: 75 : __.

? **Activity 2:** Go to the market and collect the prices of different sizes of shampoo containers of the same shampoo and create a table like the one given below. See if the volume of shampoo is proportional to the price.

Container	Volume	Price
Sachet	6 mL	₹2
Small Bottle	180 mL	₹154
Medium Bottle	340 mL	₹276
Large Bottle	1000 mL	₹540

Let us compare the ratios for the sample table above.

The ratio of the volume of a sachet to a small bottle is 6 : 180. The ratio of their prices is 2 : 154. Are these ratios proportional?

? Why do you think that the ratio of the prices is not proportional to the ratio of the volumes?

Math Talk

Discuss the pros and cons of different size bottles for the company and for customers. For reducing ecological footprint, what would you recommend to the company and to the customer?

Does the same occur for other products?

Make similar tables for other products in the market, capturing different prices for different measures of the same product, e.g., rice or *atta* (flour).

Observe the products for which the prices are proportional to the different measures.

Discuss in class the proportionality of prices to measures of the same product.

Note to the Teacher: Give a project to students. Start by dividing the class into groups. Each group should go to one shop and collect prices for different measures of the same product. For example, they should note down the prices of 500 g of rice, 1 kg of rice, and 10 kg of rice. They should make tables with measure sizes and prices, and present them to the rest of the class. They should discuss if the prices are proportional or not, and why.

7.5 Sharing, but Not Equally!

? **Activity 3:** Form a pair. Collect 12 countable objects or counters (it can be coins, seeds, or pebbles). Now, share them between the two of you in different ways.

? If you divide them equally, what is the ratio of the number of counters with each of you?

Each of you will get 6 counters. So, the ratio is 6 : 6, or 1 : 1 in its simplest form.

Now let us not share equally.

? If your partner gets 5 counters, how many objects will you get? What is the ratio of the counters?

The ratio of the counters of your partner to yours is 5 : 7.

? Now, if you want to share the counters between the two of you in the ratio of 3 : 1, how many counters would each of you get?

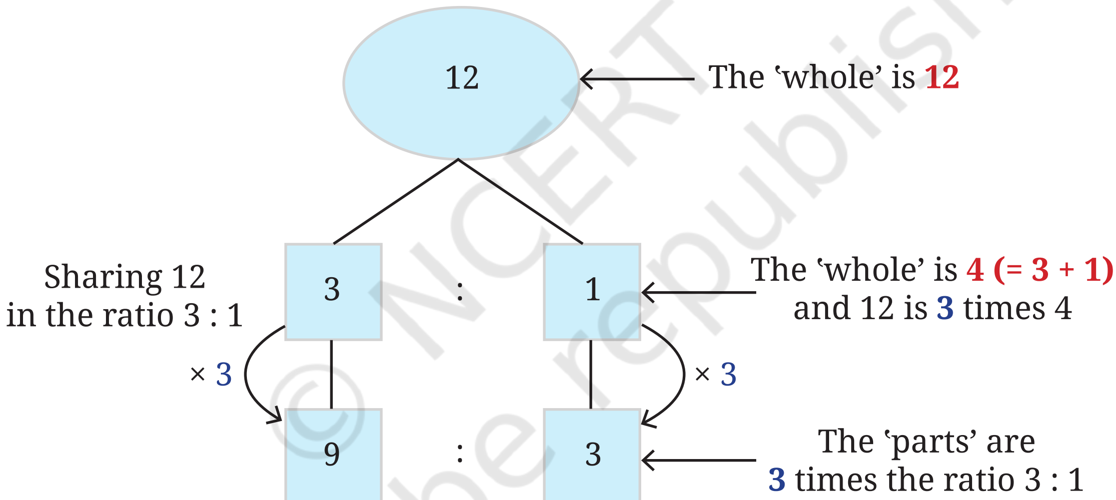
Share the counters in different ways and see which combination is in the ratio 3 : 1.

One way to share the counters in the ratio of 3 : 1 is as follows —

1. Your partner takes 3 counters and you take 1 counter. There are now 8 counters left.
2. Your partner takes 3 more counters and you take 1 more counter. There are now 4 counters left.
3. Your partner takes 3 more counters and you take 1 more counter. There are no more counters left.

So, your partner gets 9 counters in total and you get 3 counters.

When we divide 12 counters in the ratio of 3 : 1 between two people, one gets 9 counters and the other gets 3 counters.



? Now, if you want to share 42 counters between the two of you in the ratio of 4 : 3, how will you do it?

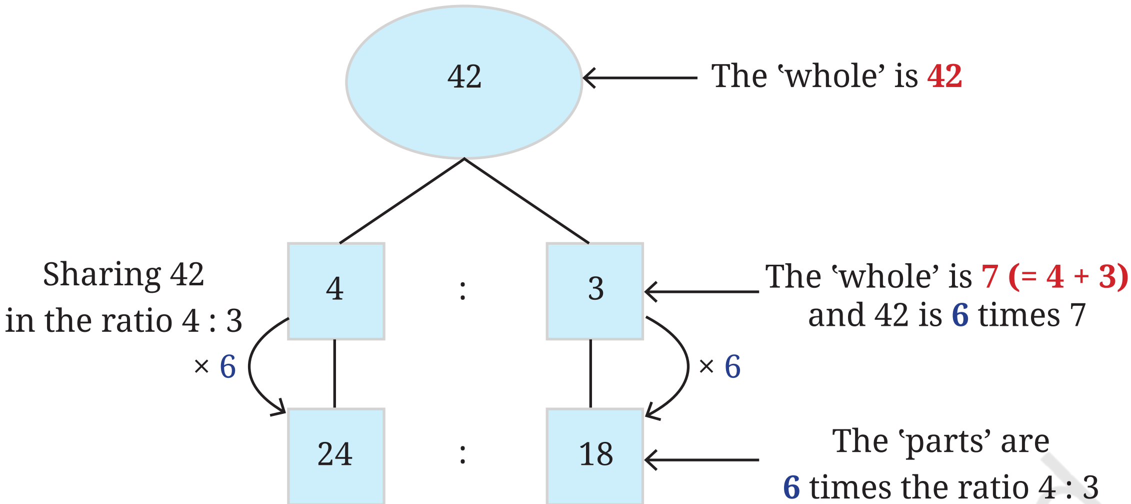
Using the same procedure would take a long time! There is a simpler way to share the whole with parts in a specified ratio.

You need to divide 42 into groups such that your partner gets 4 groups and you get 3 groups.

? What is the size of each group?

If your partner gets 4 groups and you get 3 groups, the total number of groups is 7. So, the size of each group is 42 ÷ 7 = 6.

Multiplying the number of groups by the size of each group, your partner gets 24 counters and you get 18 counters, when you share 42 counters in the ratio of 4 : 3.



In general, when we want to divide a quantity, say x, in the ratio $m : n$, we do the following:

1. We need to split x into groups such that it can be divided into two parts where the first part has m groups and the second part has n groups.

2. But what is the size of each group? This can be found out by dividing x by the number of groups. The number of groups are $m + n$. So, the size of each group is $\frac{x}{m+n}$.

3. So, the first part has $m \times \frac{x}{m+n}$ objects and the second part has $n \times \frac{x}{m+n}$ objects.

Thus, if we want to divide a quantity x in the ratio of $m : n$, then the parts will be $m \times \frac{x}{m+n}$ and $n \times \frac{x}{m+n}$. We see that

$$m \times \frac{x}{m+n} : n \times \frac{x}{m+n} :: m : n.$$

Example 11: Prashanti and Bhuvan started a food cart business near their school. Prashanti invested ₹75,000 and Bhuvan invested ₹25,000. At the end of the first month, they gained a profit of ₹4,000. They decided that they would share the profit in the same ratio as that of their investment. What is each person's share of the profit?

The ratio of their investment is 75000 : 25000.
Reducing this ratio to its simplest form, we get 3 : 1.
3 + 1 is 4 and dividing the profit of 4000 by 4, we get 1000.
So, Prashanti's share is 3 × 1000 and Bhuvan's share is 1 × 1000.
So, Prashanti would get ₹3,000 and Bhuvan would get ₹1,000 of the profit.

Example 12: A mixture of 40 kg contains sand and cement in the ratio of 3 : 1. How much cement should be added to the mixture to make the ratio of sand to cement 5 : 2?

Let us find the quantity of sand and cement in the original mixture. The ratio is 3 : 1 and the total weight is 40 kg.

So, the weight of sand is $\frac{3}{(3+1)} \times 40 = 30$ kg.

The weight of cement is $\frac{1}{(3+1)} \times 40 = 10$ kg.

The weight of sand is the same in the new mixture. It remains 30. But the new ratio of sand to cement is 5 : 2. So the question is,

$$5 : 2 :: 30 : ?$$

If the ratio is 5 : 2, then the second term is $\frac{2}{5}$ times the first term. Since the new ratio is equivalent to 5 : 2, the second term in the new ratio should also be $\frac{2}{5}$ times of 30.

$$\frac{2}{5} \times 30 = 12.$$

The new mixture should have 12 kg of cement if the ratio of sand to cement is to be 5 : 2.

There is 10 kg of cement already. So, we need to add 2 kg of cement to the original mixture.

Figure it Out

1. Divide ₹4,500 into two parts in the ratio 2 : 3.
2. In a science lab, acid and water are mixed in the ratio of 1 : 5 to make a solution. In a bottle that has 240 mL of the solution, how much acid and water does the solution contain?
3. Blue and yellow paints are mixed in the ratio of 3 : 5 to produce green paint. To produce 40 mL of green paint, how much of these two colours are needed? To make the paint a lighter shade of green, I added 20 mL of yellow to the mixture. What is the new ratio of blue and yellow in the paint?
4. To make soft idlis, you need to mix rice and *urad dal* in the ratio of 2 : 1. If you need 6 cups of this mixture to make idlis tomorrow morning, how many cups of rice and *urad dal* will you need?
5. I have one bucket of orange paint that I made by mixing red and yellow paints in the ratio of 3 : 5. I added another bucket of yellow paint to this mixture. What is the ratio of red paint to yellow paint in the new mixture?

7.6 Unit Conversions

We have noticed earlier that solving problems with proportionality often requires us to convert units from one system to another. Here are a few important unit conversions for your reference.

Length

1 metre = 3.281 feet

Area

1 square metre = 10.764 square feet

1 acre = 43,560 square feet

1 hectare = 10,000 square metres

1 hectare = 2.471 acres

Volume

1 millilitre (mL) = 1 cubic centimetre (cc)

1 litre = 1,000 mL or 1,000 cc

Temperature

Temperature conversion between Fahrenheit and Celsius is a bit more complicated. 0°C = 32 °F, and

$$\text{Fahrenheit} = \frac{9}{5} \times \text{Celsius} + 32$$

and

$$\text{Celsius} = \frac{5}{9} \times (\text{Fahrenheit} - 32)$$

For example, 25°C is 77 °F.

? Figure it Out

1. Anagh mixes 600 mL of orange juice with 900 mL of apple juice to make a fruit drink. Write the ratio of orange juice to apple juice in its simplest form.
2. Last year, we hired 3 buses for the school trip. We had a total of 162 students and teachers who went on that trip and all the buses were full. This year we have 204 students. How many buses will we need? Will all the buses be full?
3. The area of Delhi is 1,484 sq. km and the area of Mumbai is 550 sq. km. The population of Delhi is approximately 30 million and that of Mumbai is 20 million people. Which city is more crowded? Why do you say so?
4. A crane of height 155 cm has its neck and the rest of its body in the ratio 4 : 6. For your height, if your neck and the rest of the body also had this ratio, how tall would your neck be?
5. Let us try an ancient problem from Lilavati. At that time weights were measured in a unit named *palas* and *niskas* was a unit of money. "If $2\frac{1}{2}$ *palas* of saffron

costs $\frac{3}{7}$ *niskas*, O expert businessman! tell me quickly what quantity of saffron can be bought for 9 *niskas*?”

6. Harmain is a 1-year-old girl. Her elder brother is 5 years old. What will be Harmain’s age when the ratio of her age to her brother’s age is 1 : 2?
7. The mass of equal volumes of gold and water are in the ratio 37 : 2. If 1 litre of water is 1 kg in mass, what is the mass of 1 litre of gold?
8. It is good farming practice to apply 10 tonnes of cow manure for 1 acre of land. A farmer is planning to grow tomatoes in a plot of size 200 ft by 500 ft. How much manure should he buy? (Please refer to the section on Unit Conversions earlier in this chapter).
9. A tap takes 15 seconds to fill a mug of water. The volume of the mug is 500 mL. How much time does the same tap take to fill a bucket of water if the bucket has a 10-litre capacity?
10. One acre of land costs ₹15,00,000. What is the cost of 2,400 square feet of the same land?
11. A tractor can plough the same area of a field 4 times faster than a pair of oxen. A farmer wants to plough his 20-acre field. A pair of oxen takes 6 hours to plough an acre of land. How much time would it take if the farmer used a pair of oxen to plough the field? How much time would it take him if he decides to use a tractor instead?
12. The ₹10 coin is an alloy of copper and nickel called ‘cupro-nickel’. Copper and nickel are mixed in a 3 : 1 ratio to get this alloy. The mass of the coin is 7.74 grams. If the cost of copper is ₹906 per kg and the cost of nickel is ₹1,341 per kg, what is the cost of these metals in a ₹10 coin?

Try This

SUMMARY

- Ratios in the form of $a : b$ indicate that for every ‘a’ unit of the first quantity, there are ‘b’ units of the second quantity. ‘a’ and ‘b’ are the terms in the ratio.
- Two ratios—$a : b$ and $c : d$—are **proportional** (written $a : b :: c : d$) if their terms change by the same factor, i.e., if $ad = bc$.
- If x is divided into two parts in the ratio $m : n$, then the quantity of the first part is $m \times \frac{x}{m+n}$ and the quantity of the second part is $n \times \frac{x}{m+n}$.

Binairo

Binairo, also known as Takuzu, is a logic puzzle with simple rules. Binairo is generally played on a square grid with no particular size. Some cells start out filled with two symbols: here horizontal and vertical lines. The rest of the cells are empty. The task is to fill cells in such a way that:

1. Each row and each column must contain an equal number of horizontal and vertical lines.
2. More than two horizontal or vertical lines can't be adjacent.
3. Each row is unique. Each column is unique.

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Puzzle

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Solution

Solve the following Binairo puzzles:

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Dot Grid

Dot Grid